ON THE EXISTENCE OF NON-TRIVIAL INVARIANT SUBSPACES FOR CERTAIN CLASSES OF OPERATORS

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CONTENTS

INTRODUCTION

One of the fundamental open questions in the theory of linear operators is whether every bounded linear operator on a Banach space has a non-trivial invariant subspace. Even in the case of a general bounded linear operator on Hilbert space, all attempts to prove that the question has an affirmative answer have failed; and, conversely, no one has succeeded in constructing a bounded linear operator on a Banach space (or for that matter on a normed space) with no non-trivial invariant subspace.

There is a considerable literature on the invariant subspace problem and positive results have been obtained for certain classes of operators. The subject has developed in several directions. Firstly, following the work of von Neumann, Stone and others on the spectral decomposition of normal operators, where invariant subspaces play a crucial role, many papers have appeared which deal with the problem in the context of Hilbert space. In particular, as is discussed in Helson's book, "Lectures on Invariant Subspaces" , the invariant subspace problem for Hilbert space can be shown to be equivalent to a certain factorization problem in the theory of analytic functions. Secondly, there are several invariant subspace theorems which involve the notion of compactness. In 1954 Aronszajn and Smith showed that every compact operator on a Banach space has a non-trivial invariant subspace, and in

1966 Bernstein and Robinson extended this result to polynomially compact operators. Recently, some further generalizations of the Aronszajn-Smith theorem have been obtained. Thirdly, in a paper which appeared in 1952, Wermer proved an invariant subspace theorem for certain invertible operators on Banach spaces, generalizing a result of Godement.

In this thesis we are concerned with the second and third directions of progress mentioned above, and we use methods which do not depend on the existence of an inner product. Furthermore, we are able to deal with both real and complex spaces. The lay-out of the thesis is as follows. In Chapter I an outline of the theory of normed spaces and bounded linear operators is given, together with a brief discussion of the super-diagonalization of linear operators on finite-dimensional linear spaces. In Chapter II we prove firstly that polynomially compact operators on normed spaces have non-trivial invariant subspaces, and secondly that a theorem obtained by Feldman for certain quasi-nilpotent operators on complex Hilbert space generalizes to normed spaces. These results are proved in both the real and complex cases. With the former of these theorems at our disposal, in Chapter III we discuss super-diagonal forms for polynomially compact operators on real and complex spaces, and show how these are related to spectral properties of the operators concerned. This chapter derives its essential ideas from the work of Ringrose on the super-diagonalization of compact operators. In the final chapter we consider operators whose spectra lie on the unit circle, and a theorem, closely related to Wermer's result mentioned above, is proved, although by a somewhat different method. Again, we are able to deal with both real and complex spaces.

Our notation is for the most part standard, although we do not follow any particular author. We use the triplet numbering system for theorems, lemmas and definitions; i.e. (a.b.c) denotes the cth item of the bth section of Chapter a. Also, the numbering of equations is started afresh in each chapter, but not in each section.

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CHAPTER I

INTRODUCTION AND PRELIMINARY RESULTS

In this chapter we give a brief account of those parts of the theory of normed linear spaces which are relevant to the rest of the thesis. Our notation is for the most part standard, although we do not follow any particular author. However, most of the material dealt with below is to be found in one of [9,15,31].

1. Normed spaces.

Let X be a normed linear space over a field K, where K is either the reals (R) or the complexes (C). A subspace of X is a non-empty linear subset of X which will be assumed to be closed in the norm topology unless otherwise stated. A non-trivial subspace is a subspace which is different from $\{0\}$ and X. If M is a subspace of X and $x \in X$, we denote by $x + M$ the coset of M in X which contains x. Thus

$$x + M = \{x + y : y \in M\} .$$

The difference space of X modulo M is the set of cosets of M in X, endowed with the usual algebraic operations and norm. It is denoted by $X - M$.

Given $x \in X$ and a non-empty subset E of X, we define the distance from x to E, $d(x,E)$, by

$$d(x,E) = \inf \{\|x - y\| : y \in E\}.$$

We note that, if M is a subspace of X and $x \in X$, then

$$d(x,M) = \|x + M\| .$$

If, also, M is finite-dimensional, it follows from the compactness of closed, bounded subsets of M that there exists $y \in M$ such that

$$d(x,M) = \|x - y\| .$$

Such a y will be called a <u>nearest point of</u> M <u>to</u> x.

Suppose that M and N are finite-dimensional subspaces of X, and that $M \subseteq N$, $M \neq N$. The canonical mapping of N on to $N - M$ has norm 1, and this norm is attained. Hence, there exists $u \in N$ such that

$$\|u\| = \|u + M\| = d(u,M) = 1 .$$

We shall call such a u a <u>unit vector in</u> N <u>orthogonal to</u> M.

The set of all bounded linear operators on X will be denoted by $B(X)$, and the identity operator on X by I. Given T in $B(X)$, a subspace M of X is <u>invariant</u> for T if $T(M) \subseteq M$. We remark that the symbol $\subseteq$ will be used for set inclusion (with the possibility of equality), and $\subset$ will be reserved for strict inclusion. Thus, a non-trivial invariant subspace for T is a closed linear subset M of X such that

$$\text{(i)} \quad T(M) \subseteq M ,$$

$$\text{(ii)} \quad \{0\} \subset M \subset X .$$

It is an open question whether every bounded linear operator on a normed space has a non-trivial invariant subspace. By restricting attention to certain classes of operators (e.g. compact operators), positive results have been obtained, but even in the case of a general operator on separable Hilbert space, the question remains open. It should perhaps be remarked that, trivially, the problem has an affirmative solution when X is a non-separable normed space. For, if $0 \neq e \in X$, and if M is the closed linear span of $\{e, Te, T^2e, \ldots\}$, then it is clear that M is a non-trivial invariant subspace for T.

Suppose that T is in $B(X)$, and that M is an invariant subspace for T. We define linear operators $T|M$ and T_M, on M and $X - M$ respectively, by

$$(T|M)x = Tx \qquad (x \in M),$$

$$T_M(x + M) = Tx + M \qquad (x + M \in X - M).$$

It is easy to check that these are well-defined, and that $T|M \in B(M)$ and $T_M \in B(X - M)$.

Given T in $B(X)$, where X is a complex normed space, the spectrum of T will be denoted by $Sp(T)$; i.e.

$$Sp(T) = \{\lambda \in C : (\lambda I - T) \text{ is not invertible in } B(X)\}.$$

The spectral radius, $r(T)$, of T is given by

$$r(T) = \inf_{n \geq 1} \|T^n\|^{1/n} = \lim_{n \to \infty} \|T^n\|^{1/n}.$$

If X is a Banach space, then $Sp(T)$ is a compact subset of C; and

$$r(T) = \max\{|\lambda| : \lambda \in Sp(T)\}.$$

Now, suppose that X is a real normed space. We denote by X_c the complexification of X; i.e.

$$X_c = \{(x,y) : x,y \in X\},$$

with

$$(x,y) + (u,v) = (x + u, y + v),$$

$$(\alpha + i\beta)(x,y) = (\alpha x - \beta y, \alpha y + \beta x),$$

$$\|(x,y)\| = \frac{1}{\sqrt{2}} \sup_\theta \{\|x \cos\theta - y \sin\theta\| + \|x \sin\theta + y \cos\theta\|\},$$

for $x,y,u,v \in X$ and $\alpha, \beta \in R$. Then X_c is a complex normed space, and $X_o = \{(x,0) : x \in X\}$ is a real subspace of X_c, which is isometrically isomorphic to X. We shall identify X with X_o and write $(x,y) = x + iy$. Then X is embedded in its complexification X_c, and $X_c = X \oplus iX$. (See [15], pp.150-152.) Also, X_c is a Banach space if and only if X is a Banach space. Let $T \in B(X)$, and define $T_c : X_c \longrightarrow X_c$ by

$$T_c(x + iy) = Tx + iTy \qquad (x + iy \in X_c) .$$

Then $T_c \in B(X_c)$, and $\|T_c\| = \|T\|$. The spectrum of T is defined by

$$Sp(T) = Sp(T_c) .$$

$Sp(T)$ is a self-conjugate subset of C. As in the complex case, the spectral radius of T is defined by

$$r(T) = \inf_{n \geq 1} \|T^n\|^{1/n} = \lim_{n \to \infty} \|T^n\|^{1/n} ;$$

and it is easy to verify that $r(T) = r(T_c)$. If X is a Banach space, then $Sp(T)$ is a compact subset of C, and

$$r(T) = \max \{|\lambda| : \lambda \in Sp(T)\} .$$

We conclude this section by giving some notation. Let E be a non-empty subset of a normed space X (over K). We denote by $cl(E)$ the closure of E in the norm topology, and by $[E]$ the linear span of E.

2. Finite-dimensional spaces.

We now review some standard results on the super-diagonalization of linear operators on finite-dimensional linear spaces. We treat the real and complex cases separately, since the theorems take slightly different forms in each case.

The following theorem for complex spaces is essentially Theorem 1, [15] p.106.

(1.2.1) Theorem. Let T be a linear operator on an n-dimensional complex linear space X, $(n \geq 1)$. Then there exists a nest $\{M_j\}_{j=0}^{n}$ of subspaces of X such that

$$\text{(i)} \quad \{0\} = M_0 \subset M_1 \subset \cdots\cdots \subset M_n = X ,$$

$$\text{(ii)} \quad \dim M_j = j \qquad (0 \leq j \leq n) ,$$

$$\text{(iii)} \quad T(M_j) \subseteq M_j \qquad (0 \leq j \leq n) .$$

Further, for $1 \leq j \leq n$, let $e_j \in M_j \setminus M_{j-1}$; and suppose that

$$Te_j = \alpha_j e_j + x_j ,$$

where $x_j \in M_{j-1}$ and $\alpha_j \in C$. Then $\{\alpha_j : 1 \leq j \leq n\}$ is the set of eigenvalues of T, each repeated to the appropriate algebraic multiplicity.

The proof of the first part of the above theorem follows easily from the fact that every linear operator on a finite-dimensional complex linear space has an eigenvalue. From this result, it can be seen that every linear operator on a finite-dimensional complex linear space of dimension greater than 1 has a non-trivial invariant subspace. Corresponding to this, in the real case we have the following lemma.

(1.2.2) Lemma. Let T be a linear operator on a real linear space X of finite dimension greater than 2. Then T has a non-trivial invariant subspace.

Proof. Let T_c be the linear operator induced by T on the complexification, X_c, of X. Suppose that $\alpha + i\beta$ is an eigenvalue of T_c, with corresponding eigenvector $e + if$. Then the linear span, $[e,f]$, of $\{e,f\}$ in X is a non-trivial invariant subspace for T .

It is well known that there are linear operators on two-dimensional real linear spaces which do not have non-trivial invariant subspaces. We shall call such operators <u>irreducible</u>.

(1.2.3) <u>Lemma</u>. Let T be an irreducible linear operator on a two-dimensional real linear space X . Then there exists a basis $\{e,f\}$ of X such that, for some α, β in R ,

$$Te = \alpha e - \beta f ,$$

$$Tf = \beta e + \alpha f .$$

Further, $\beta \neq 0$; and $\alpha + i\beta$, $\alpha - i\beta$ are the eigenvalues of T_c , the corresponding operator on X_c .

<u>Proof</u>. Let $\alpha + i\beta$ be an eigenvalue of T_c , with corresponding eigenvector $e + if$. Then $\alpha - i\beta$ is an eigenvalue of T_c , corresponding to the eigenvector $e - if$. It is elementary to check that Te and Tf have the required form. Also, the irreducibility of T implies that $\{e,f\}$ is a basis for X , and that $\beta \neq 0$.

The analogue for real spaces of Theorem (1.2.1) can now be stated. No suitable reference could be found in the literature, but the proof follows easily from Lemmas (1.2.2) and (1.2.3).

(1.2.4) <u>Theorem</u>. Let T be a linear operator on an n-dimensional real linear space X , $(n \geq 1)$. Then there exists a nest $\{M_j\}_{j=0}^{m}$ of subspaces of X such that

(i) $\{0\} = M_o \subset M_1 \subset \dots \subset M_m = X$,

(ii) $\dim (M_j - M_{j-1}) \leq 2 \qquad (1 \leq j \leq m)$,

(iii) $T(M_j) \subseteq M_j \qquad (0 \leq j \leq m)$,

(iv) if $\dim (M_j - M_{j-1}) = 2$, then the operator $(T|M_j)_{M_{j-1}}$ induced by T on $M_j - M_{j-1}$ is irreducible.

Let $\{M_{j_r}\}_{r=1}^{k}$ be the set of M_j's such that $\dim (M_j - M_{j-1}) = 1$; and let $e_r \in M_{j_r} \setminus M_{j_r-1}$ and $\alpha_r \in R$ be such that

$$Te_r - \alpha_r e_r \in M_{j_r-1} ,$$

for $1 \leq r \leq k$.

Let $\{M_{j_r}\}_{r=k+1}^{m}$ be the set of M_j's such that $\dim (M_j - M_{j-1}) = 2$. We can choose e_r, f_r in M_{j_r} such that $\{e_r + M_{j_r-1}, f_r + M_{j_r-1}\}$ is a basis for $M_{j_r} - M_{j_r-1}$, and such that, for some α_r, β_r in R ,

$$Te_r - (\alpha_r e_r - \beta_r f_r) \in M_{j_r-1} ,$$

$$Tf_r - (\beta_r e_r + \alpha_r f_r) \in M_{j_r-1} ,$$

for $k + 1 \leq r \leq m$.

Then

$$\{\alpha_r : 1 \leq r \leq k\} \cup \{\alpha_r \pm i\beta_r : k + 1 \leq r \leq m\}$$

is the set of eigenvalues of T_c , each repeated to the appropriate algebraic multiplicity.

We conclude this section with a result which is not of great interest in itself, but which will be of some technical use in Chapter II. It is an immediate consequence of Theorems (1.2.1) and (1.2.4).

(1.2.5) <u>Theorem</u>. Let T be a linear operator on an n-dimensional linear space X over K , $(n \geq 1)$. Then there exists a nest $\{M_j\}_{j=0}^{m}$ of subspaces of X such that

(i) $\{0\} = M_o \subset M_1 \subset \cdots\cdot \subset M_m = X$,

(ii) $\dim (M_j - M_{j-1}) \leq 2 \qquad (1 \leq j \leq m)$,

(iii) $T(M_j) \subseteq M_j \qquad (0 \leq j \leq m)$.

CHAPTER II

COMPACT OPERATORS

An important class of operators with non-trivial invariant subspaces is the class of compact operators. This was first proved by Aronszajn and Smith, [1], who showed that every compact operator on a complex Banach space of dimension greater than 1 has a non-trivial invariant subspace. (In fact, the corresponding result for Hilbert space had already been proved by von Neumann and Aronszajn.) As remarked by Bonsall in [5], where he gives a simplified version of their proof, the completeness of the underlying space is not necessary for the theorem of Aronszajn and Smith to hold.

After the Aronszajn-Smith theorem appeared, the question was raised whether an operator with the property that its square is compact has an non-trivial invariant subspace. This problem was solved by Bernstein and Robinson, [4,26 Chapter 7], who showed that if H is a complex Hilbert space, and $T \in B(H)$ is such that $p(T)$ is compact for some non-zero polynomial p, then T has a non-trivial invariant subspace. Bernstein, [3], has also shown that this result holds in complex Banach spaces. Bernstein and Robinson use methods from non-standard analysis, but in fact their results can be obtained within the usual framework of functional analysis. In [16], Halmos gives a proof of their theorem in Hilbert space, and in [5] Bonsall gives a proof of the theorem in complex normed spaces. Both of these proofs use standard methods only.

Recently, several papers have appeared, [2,7,11], in which the methods and techniques initiated by Aronszajn and Smith are developed. These papers deal with Hilbert space, and use as a principal tool sequences of orthogonal projections of finite rank. However, it appears that one can obtain similar results for normed spaces by considering, instead, sequences of finite-dimensional subspaces. Our intention in this chapter is to prove the Bernstein-Robinson theorem and a theorem of Feldman, [2,11], in the context of normed spaces, using the techniques developed by Bonsall in [5]. We shall add one further idea which will enable us to deal with the real and complex cases simultaneously.

1. Linear operators with cyclic vectors.

Throughout this section X will be an infinite-dimensional normed linear space over K.

(2.1.1) Definition. Given T in $B(X)$, a vector e in X is cyclic for T if

$$X = cl[e, Te, T^2e, \dots].$$

Remark. If e is a cyclic vector for T, then $\{e, Te, T^2e, \dots\}$ is a linearly independent set. For, if $T^n e \in [e, Te, T^2e, \dots, T^{n-1}e]$ for some $n \geq 1$, (taking $T^0 = I$), then $T^m e \in [e, Te, T^2e, \dots, T^{n-1}e]$ for all $m \geq n$. Hence $X = [e, Te, T^2e, \dots, T^{n-1}e]$ and is finite-dimensional, contradicting the assumption on the dimension of X. In particular, $e \neq 0$, and so, by a suitable normalization, we can take $\|e\| = 1$.

For the rest of this section, we shall consider a fixed T in $B(X)$ with a fixed cyclic vector e of norm 1. Let $X_0 = \{0\}$, and

$$X_n = [e, Te, \dots, T^{n-1}e]$$

for $n \geq 1$. By the linear independence of $\{e, Te, T^2e, \dots\}$, $\dim X_n = n$;

and

$$[e] = X_1 \subset X_2 \subset \dots \subset X_n \subset \dots \ .$$

For $n \geq 1$, let e_n be a unit vector in X_n orthogonal to X_{n-1} ; i.e.

$$\|e_n\| = 1 = d(e_n, X_{n-1}) \qquad (n \geq 1) .$$

Since $X_n = [X_{n-1}, e_n]$ and $T(X_{n-1}) \subseteq X_n$, we can define linear operators $T_n : X_n \longrightarrow X_n$ for $n \geq 1$ by

$$T_n x = Tx \qquad (x \in X_{n-1})$$

$$T_n e_n = u_n ,$$

where u_n is a nearest point of X_n to Te_n .

(2.1.2) Lemma. If $x \in X_n$ and $n \geq 1$, then

$$d(Tx, X_n) = \|Tx - T_n x\| .$$

Proof. Let $x \in X_n$. Then $x = \lambda e_n + y$ for some $\lambda \in K$ and $y \in X_{n-1}$. Thus

$$T_n x = \lambda u_n + Ty , \ Tx = \lambda Te_n + Ty .$$

Hence

$$\begin{aligned} \|Tx - T_n x\| &= |\lambda| \|u_n - Te_n\| \\ &= |\lambda| d(Te_n, X_n) \\ &= d(\lambda Te_n, X_n) . \end{aligned}$$

Now, $Ty \in X_n$; and so

$$d(Tx, X_n) = d(\lambda Te_n + Ty, X_n) = d(\lambda Te_n, X_n) .$$

Hence

$$\|Tx - T_n x\| = d(Tx, X_n) .$$

(2.1.3) Lemma. For each integer $k \geq 1$, there is a constant $M_k > 0$, independent of n, such that

$$\|T^k x - T_n^k x\| \leq M_k d(Te_n, X_n) \|x\|$$

for $x \in X_n$, $n \geq 1$, $k \geq 1$.

Proof. By induction on k .

Suppose that $k = 1$. Let $x = \lambda e_n + y \in X_n$, where $\lambda \in K$ and $y \in X_{n-1}$. From the proof of Lemma (2.1.2),

$$\|Tx - T_n x\| = |\lambda| d(Te_n, X_n) .$$

Also,

$$|\lambda| = |\lambda| d(e_n, X_{n-1}) = d(\lambda e_n, X_{n-1})$$
$$= d(x, X_{n-1}) \leq \|x\| .$$

Thus,

$$\|Tx - T_n x\| \leq d(Te_n, X_n)\|x\| ;$$

and we can take $M_1 = 1$.

Suppose now that $k > 1$, and that the result holds for $k - 1$. Let $x \in X_n$.

$$\|T^k x - T_n^k x\| \leq \|T^k x - TT_n^{k-1} x\| + \|TT_n^{k-1} x - T_n^k x\|$$
$$\leq \|T\|\|T^{k-1} x - T_n^{k-1} x\| + \|TT_n^{k-1} x - T_n^k x\| .$$

By the induction hypothesis,

$$\|T^{k-1} x - T_n^{k-1} x\| \leq M_{k-1} d(Te_n, X_n)\|x\| .$$

Also, by the case $k = 1$,

$$\|TT_n^{k-1} x - T_n^k x\| \leq d(Te_n, X_n)\|T_n^{k-1} x\| .$$

Now

$$\|T_n^{k-1} x\| \leq \|T_n^{k-1} x - T^{k-1} x\| + \|T^{k-1} x\|$$
$$\leq \{M_{k-1} d(Te_n, X_n) + \|T^{k-1}\|\}\|x\|$$
$$\leq \{M_{k-1}\|T\| + \|T^{k-1}\|\}\|x\| .$$

Hence

$$\|T^k x - T_n^k x\| \leq M_k d(Te_n, X_n)\|x\| ,$$

where $M_k = 2M_{k-1}\|T\| + \|T^{k-1}\|$, and the result is established by induction.

The following corollary is easily obtained from this result.

(2.1.4) <u>Corollary</u>. Given a polynomial p with coefficients in K, there is a constant $M > 0$ such that

$$\|p(T)x - p(T_n)x\| \leq M\, d(Te_n, X_n)\|x\| ,$$

for $x \in X_n$ and $n \geq 1$.

(2.1.5) <u>Definition</u>. Given a sequence $\{E_n\}_{n=1}^{\infty}$ of subspaces of X, define $\liminf E_n$ to be the set of all $x \in X$ such that there exists a sequence $\{x_n\}_{n=1}^{\infty}$ with $x_n \in E_n$ and $x_n \longrightarrow x$ as $n \rightarrow \infty$.

It is clear that $\liminf E_n$ is a (closed) linear subspace of X, and that

$$\liminf E_n = \{x \in X : d(x, E_n) \longrightarrow 0 \text{ as } n \rightarrow \infty\} .$$

(2.1.6) <u>Lemma</u>. Let $\{n_k\}_{k=1}^{\infty}$ be a subsequence of $\{1,2,3,..\}$, and for $k \geq 1$ let E_k be a subspace of X_{n_k} such that $T_{n_k}(E_k) \subseteq E_k$. Then

(i) $\liminf T(E_k) \subseteq \liminf E_k$,

(ii) $\liminf E_k$ is invariant for T.

<u>Proof</u>. (i) Let $x \in \liminf T(E_k)$. Then there exists $y_k \in E_k$ such that

$$\|x - Ty_k\| = d(x, T(E_k)) \longrightarrow 0$$

as $k \rightarrow \infty$. We have

$$\begin{aligned} \|x - T_{n_k}y_k\| &\leq \|x - Ty_k\| + \|Ty_k - T_{n_k}y_k\| \\ &= \|x - Ty_k\| + d(Ty_k, X_{n_k}), \text{ by Lemma (2.1.2)} \\ &\leq 2\|x - Ty_k\| + d(x, X_{n_k}). \end{aligned}$$

$X = \mathrm{cl}\{\bigcup_{k=1}^{\infty} X_{n_k}\}$, and so $d(x, X_{n_k}) \rightarrow 0$ as $k \rightarrow \infty$. Also,

$$\|x - Ty_k\| \longrightarrow 0$$

as $k \rightarrow \infty$. Hence,

$$\|x - T_{n_k}y_k\| \longrightarrow 0$$

as $k \to \infty$. But $T_{n_k} y_{n_k} \in T_{n_k}(E_k) \subseteq E_k$, and so

$$d(x, E_k) \leq \|x - T_{n_k} y_k\| \longrightarrow 0$$

as $k \to \infty$. Therefore $x \in \liminf E_k$.

(ii) Let $x \in \liminf E_k$. Then, by the continuity of T, $Tx \in \liminf T(E_k)$. Hence, by (i), $Tx \in \liminf E_k$. Therefore, $\liminf E_k$ is invariant for T.

Thus, we have a method of constructing invariant subspaces for T since, by Theorem (1.2.5), we can certainly choose subspaces E_n of X_n invariant for T_n. The difficulty lies in the fact that we must do this in such a way that $\liminf E_{n_k}$ is non-trivial for some sequence $\{n_k\}$. In the next two sections we use a device of Aronszajn and Smith to do this for certain types of operators, and obtain two invariant subspace theorems.

2. The Bernstein-Robinson theorem.

In this section we prove that, if X is a normed space and T a bounded linear operator on X which is polynomially compact, then T has a non-trivial invariant subspace. The proof is essentially that given in [5], but we shall deal with both real and complex spaces. Before proceeding, we require a lemma on compact operators.

(2.2.1) Lemma. Let X be a normed linear space over K, and let S be a compact linear operator on X. Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of subsets of X such that

$$\text{(i)}\quad X_n \subseteq X_{n+1} \qquad (n \geq 1),$$
$$\text{(ii)}\quad X = \mathrm{cl}\{\bigcup_{n=1}^{\infty} X_n\}.$$

Then $d(Sx, X_n) \longrightarrow 0$ as $n \to \infty$, uniformly for $\|x\| \leq 1$.

Proof. Let $E = \mathrm{cl}\{Sx : \|x\| \leq 1\}$. Then E is compact in the norm topology. For $n \geq 1$, define functions $f_n : E \longrightarrow R$ by

$$f_n(x) = d(x, X_n) \qquad (x \in E) .$$

Then f_n is continuous with respect to the norm topology; and, since $X = \mathrm{cl}\{\bigcup_{n=1}^{\infty} X_n\}$, $f_n \downarrow 0$ pointwise on E . Hence, by Dini's theorem, $f_n \downarrow 0$ uniformly on E . Therefore

$$f_n(Sx) = d(Sx, X_n) \longrightarrow 0$$

as $n \longrightarrow \infty$, uniformly for $\|x\| \leq 1$.

(2.2.2) Theorem. Let X be a normed linear space over C (resp. R) of dimension greater than 1 (resp. 2) , and let $T \in B(X)$. Suppose that p is a non-zero polynomial with coefficients in C (resp. R) such that $p(T)$ is compact. Then T has a non-trivial invariant subspace.

Proof. From the remarks following Theorem (1.2.1) and from Lemma (1.2.2), we can assume that X is infinite-dimensional. Put $K = C$ or R according as X is complex or real. Let $e \in X$ with $\|e\| = 1$. We can assume that e is a cyclic vector for T since, otherwise, $\mathrm{cl}[e, Te, T^2e, \ldots]$ is a non-trivial invariant subspace for T . We can also assume that $e, p(T)e, \{p(T)\}^2e, \ldots$ are linearly independent; for, otherwise, we easily obtain a non-trivial invariant subspace as follows. Let $S = p(T)$. Suppose that, for some $n \geq 1$, $e, Se, S^2e, \ldots, S^{n-1}e$ are linearly independent, but $e, Se, S^2e, \ldots, S^ne$ are linearly dependent. Then there exist $\lambda_0, \lambda_1, \ldots, \lambda_n$ in K , with $\lambda_n \neq 0$, such that

$$\lambda_0 e + \lambda_1 Se + \ldots + \lambda_n S^n e = 0 .$$

Since $ST = TS$,

$$\{x : (\lambda_0 I + \lambda_1 S + \ldots + \lambda_n S^n)x = 0\}$$

is an invariant subspace for T , and is non-trivial unless

$$\lambda_0 I + \lambda_1 S + \ldots + \lambda_n S^n = 0 .$$

If this is the case, then $\lambda_0 = 0$, since S is compact and X is infinite-dimensional. This gives

$$S(\lambda_1 e + \ldots + \lambda_n S^{n-1} e) = 0 .$$

By the linear independence of $e, Te, \ldots$, $Se \neq 0$; and, by the linear independence of $e, Se, \ldots, S^{n-1}e$, $\lambda_1 e + \ldots + \lambda_n S^{n-1} e \neq 0$. Hence

$$\{x : Sx = 0\}$$

is a non-trivial invariant subspace for T . Thus we can assume that $e, p(T)e, \{p(T)\}^2 e, \ldots$ are linearly independent. Finally, without loss of generality we take the leading coefficient of p to be 1 , and suppose that p is of degree r .

Since e is a cyclic vector for T , we can define X_n , e_n and T_n as in section 1 . For $n \geq 1$,

$$[X_{n-1}, e_n] = [X_{n-1}, T^{n-1}e] = X_n ;$$

and so

$$e_n - \alpha_n T^{n-1} e \in X_{n-1} \tag{1}$$

for some α_n in K , $\alpha_n \neq 0$. Since $T(X_{n-1}) \subseteq X_n$, from (1) we get

$$Te_n - \alpha_n T^n e \in X_n ; \tag{2}$$

and, replacing n by $n + 1$ in (1),

$$e_{n+1} - \alpha_{n+1} T^n e \in X_n . \tag{3}$$

From (2) and (3),

$$Te_n - \frac{\alpha_n}{\alpha_{n+1}} e_{n+1} \in X_n ;$$

and so

$$d(Te_n, X_n) = \frac{|\alpha_n|}{|\alpha_{n+1}|} d(e_{n+1}, X_n) = \frac{|\alpha_n|}{|\alpha_{n+1}|} . \tag{4}$$

Also, from (1),

$$T^r e_n - \alpha_n T^{n+r-1} e \in X_{n+r-1} ,$$

and

$$e_{n+r} - \alpha_{n+r} T^{n+r-1} e \in X_{n+r-1} .$$

Hence

$$T^r e_n - \frac{\alpha_n}{\alpha_{n+r}} e_{n+r} \in X_{n+r-1} . \tag{5}$$

Now $p(T)$ has leading coefficient 1 , and $T^{r'} e_n \in X_{n+r-1}$ if $r' < r$. Therefore it follows from (5) that

$$p(T)e_n - \frac{\alpha_n}{\alpha_{n+r}} e_{n+r} \in X_{n+r-1} .$$

Thus

$$\begin{aligned} d(p(T)e_n, X_{n+r-1}) &= \frac{|\alpha_n|}{|\alpha_{n+r}|} d(e_{n+r}, X_{n+r-1}) = \frac{|\alpha_n|}{|\alpha_{n+r}|} \\ &= \prod_{m=n}^{n+r-1} \frac{|\alpha_m|}{|\alpha_{m+1}|} . \end{aligned}$$

Then, by (4),

$$d(p(T)e_n, X_{n+r-1}) = \prod_{m=n}^{n+r-1} d(Te_m, X_m) . \tag{6}$$

Now, $X = \mathrm{cl}\{\bigcup_{n=1}^{\infty} X_{n+r-1}\}$ and $p(T)$ is compact. Therefore, by Lemma (2.2.1), since $\|e_n\| = 1$,

$$d(p(T)e_n, X_{n+r-1}) \longrightarrow 0$$

as $n \longrightarrow \infty$. Hence, by (6), there is a sequence $\{j(n)\}$ such that

$$d(Te_{j(n)}, X_{j(n)}) \longrightarrow 0 \tag{7}$$

as $n \longrightarrow \infty$.

By Theorem (1.2.5), we can find sequences $\{X_n^i\}_{i=0}^{m_n}$ of subspaces of $X_{j(n)}$ such that

$$\text{(i)} \quad \{0\} = X_n^0 \subset X_n^1 \subset \dots \subset X_n^{m_n} = X_{j(n)} \qquad (n \geq 1) ,$$

(ii) $\dim(X_n^i - X_n^{i-1}) \leq 2$ $\qquad (1 \leq i \leq m_n\ ,\ n \geq 1)\ ,$

(iii) $T_{j(n)}(X_n^i) \subseteq X_n^i$ $\qquad (0 \leq i \leq m_n\ ,\ n \geq 1)\ .$

Since $e, p(T)e, \ldots$ are linearly independent, $p(T)e \neq 0$; and so we can choose α such that $0 < \alpha < 1$ and

$$\|p(T)e\| > \alpha \|p(T)\| \ . \tag{8}$$

For each $n \geq 1$, $d(e, X_n^0) = \|e\| = 1$ and $d(e, X_n^{m_n}) = 0$. Hence, there is a greatest integer, i_n say, such that $d(e, X_n^{i_n}) > \alpha$. Let $F_n = X_n^{i_n}$ and $G_n = X_n^{i_n+1}$. Then $F_n \subset G_n$, $\dim(G_n - F_n) \leq 2$ and

$$d(e, F_n) > \alpha \ , \ d(e, G_n) \leq \alpha \ . \tag{9}$$

It follows from (9) that, for every subsequence $\{n_k\}$, $e \notin \liminf F_{n_k}$. Therefore, by Lemma (2.1.6) (ii), $\liminf F_{n_k}$ is a non-trivial invariant subspace for T , unless

$$\liminf F_{n_k} = \{0\} \ . \tag{10}$$

We shall assume that (10) holds for all subsequences $\{n_k\}$, and show that in this case $\liminf G_{n_k}$ is non-trivial for some subsequence $\{n_k\}$. This will complete the proof since, again by Lemma (2.1.6) (ii), $\liminf G_{n_k}$ is invariant for T .

There are two possible cases to be considered.

<u>Case (a)</u>. $\dim(G_n - F_n) = 1$ for infinitely many n . Then, by passing to a subsequence if necessary, we can assume that $\dim(G_n - F_n) = 1$ for all n . Let u_n be a unit vector in G_n orthogonal to F_n , and let x_n , x_n' be nearest points of G_n to e , $p(T)e$ respectively; i.e.

$$\|e - x_n\| = d(e, G_n) \ , \ \|p(T)e - x_n'\| = d(p(T)e, G_n) \ . \tag{11}$$

Since $\dim(G_n - F_n) = 1$,

$$G_n = [u_n, F_n] \ ;$$

and so we can write

$$\begin{aligned} x_n &= \lambda_n u_n + y_n \\ x_n' &= \lambda_n' u_n + y_n' \end{aligned} \qquad (n \geq 1) \qquad (12)$$

where y_n , $y_n' \in F_n$ and λ_n, $\lambda_n' \in K$.

$$|\lambda_n| = d(x_n, F_n) \leq \|x_n\| \leq \|e - x_n\| + \|e\|$$
$$= d(e, G_n) + \|e\| \leq 2\|e\| \ .$$

Hence $\{|\lambda_n|\}$ and $\{\|x_n\|\}$ are bounded sequences . Similarly, $\{|\lambda_n'|\}$ is a bounded sequence. Thus, by the compactness of $p(T)$, we can find a subsequence $\{n_k\}$ such that

$$\lambda_{n_k} \longrightarrow \lambda, \quad \lambda_{n_k}' \longrightarrow \lambda', \ p(T)x_{n_k} \longrightarrow x$$

as $k \longrightarrow \infty$, where λ, $\lambda' \in K$ and $x \in X$.

We show that $x \in \liminf G_{n_k}$. By Corollary (2.1.4) and the fact that $\{\|x_n\|\}$ is bounded, there is a constant $M > 0$ such that

$$\|p(T)x_{n_k} - p(T_{j(n_k)})x_{n_k}\| \leq M\, d(Te_{j(n_k)}, X_{j(n_k)}) \qquad (k \geq 1) \ .$$

Then, by (7),

$$\|p(T)x_{n_k} - p(T_{j(n_k)})x_{n_k}\| \longrightarrow 0$$

as $k \longrightarrow \infty$. But $p(T)x_{n_k} \longrightarrow x$; and so

$$p(T_{j(n_k)})x_{n_k} \longrightarrow x$$

as $k \longrightarrow \infty$. Since $T_{j(n_k)}(G_{n_k}) \subseteq G_{n_k}$, $p(T_{j(n_k)})x_{n_k} \in G_{n_k}$; and it now follows that

$$x \in \liminf G_{n_k} \ .$$

For $n \geq 1$,

$$\begin{aligned} \|p(T)x_n\| &\geq \|p(T)e\| - \|p(T)e - p(T)x_n\| \\ &\geq \|p(T)e\| - \|p(T)\|\|e - x_n\| \\ &= \|p(T)e\| - \|p(T)\|d(e, G_n) \\ &\geq \|p(T)e\| - \alpha\|p(T)\| \ , \text{ by } (9) \ . \end{aligned}$$

Therefore, by (8),

$$\|x\| \geq \|p(T)e\| - \alpha\|p(T)\| > 0 ;$$

and so $x \neq 0$. But $x \in \liminf G_{n_k}$, and hence

$$\liminf G_{n_k} \neq \{0\} .$$

Finally, we show that $\liminf G_{n_k} \neq X$, and this completes the proof for case (a). Suppose that $\liminf G_{n_k} = X$. Then, by (11),

$$x_{n_k} \longrightarrow e \quad \text{and} \quad x'_{n_k} \longrightarrow p(T)e$$

as $k \longrightarrow \infty$. From (12), we get that

$$\lambda'_{n_k} y_{n_k} - \lambda_{n_k} y'_{n_k} = \lambda'_{n_k} x_{n_k} - \lambda_{n_k} x'_{n_k} \longrightarrow \lambda' e - \lambda p(T)e$$

as $k \longrightarrow \infty$; and so

$$\lambda' e - \lambda p(T)e \in \liminf F_{n_k} = \{0\} , \text{ by } (10) .$$

Therefore

$$\lambda = \lambda' = 0 ,$$

by the linear independence of $e, p(T)e$. Hence, by (12),

$$x_{n_k} - y_{n_k} = \lambda_{n_k} u_{n_k} \longrightarrow 0$$

as $k \longrightarrow \infty$; and so

$$y_{n_k} \longrightarrow e$$

as $k \longrightarrow \infty$, since $x_{n_k} \longrightarrow e$. This gives

$$e \in \liminf F_{n_k} = \{0\} ,$$

an obvious contradiction. Thus $\liminf G_{n_k} \neq X$, and the proof is complete in case (a) .

<u>Case (b)</u>. $\dim (G_n - F_n) = 1$ for only a finite number of n . Then, by omitting these, we can assume that $\dim (G_n - F_n) = 2$ for all n . Let u_n be a unit vector in G_n orthogonal to F_n . Since $\dim (G_n - F_n) = 2$, $G_n \neq [u_n, F_n]$. Hence there exists a unit vector v_n in G_n orthogonal to $[u_n, F_n]$. Thus

$$\|u_n\| = \|v_n\| = d(u_n, F_n) = d(v_n, [u_n, F_n]) = 1 .$$

Let x_n , x_n' , w_n , w_n' be nearest points of G_n to e , $p(T)e$, , $\{p(T)\}^2 e$, $\{p(T)\}^3 e$ respectively. Since $\dim(G_n - F_n) = 2$, $G_n = [u_n, v_n, F_n]$; and so we can write

$$x_n = \lambda_n u_n + \mu_n v_n + y_n \tag{13}$$

$$x_n' = \lambda_n' u_n + \mu_n' v_n + y_n' \tag{14}$$

$$w_n = \theta_n u_n + \phi_n v_n + z_n \tag{15}$$

$$w_n' = \theta_n' u_n + \phi_n' v_n + z_n' \tag{16}$$

where λ_n , λ_n' , μ_n , μ_n' , θ_n , θ_n' , ϕ_n , $\phi_n' \in K$ and y_n , y_n' , z_n , $z_n' \in F_n$.

$$|\mu_n| = d(x_n, [u_n, F_n]) \leq \|x_n\| \leq \|x_n - e\| + \|e\|$$
$$= d(e, G_n) + \|e\| \leq 2\|e\| \ .$$

Also,

$$|\lambda_n| = d(x_n - \mu_n v_n, F_n) \leq \|x_n - \mu_n v_n\|$$
$$\leq \|x_n\| + |\mu_n| \leq 4\|e\| \ .$$

Thus $\{\|x_n\|\}$, $\{|\lambda_n|\}$, and $\{|\mu_n|\}$ are bounded sequences. Similarly, $\{|\lambda_n'|\}$, $\{|\mu_n'|\}$, $\{|\theta_n|\}$, $\{|\phi_n|\}$, $\{|\theta_n'|\}$, and $\{|\phi_n'|\}$ are bounded sequences. Hence, using the compactness of $p(T)$, we can choose a subsequence $\{n_k\}$ such that

$$p(T)x_{n_k} \longrightarrow x \ ,$$
$$\lambda_{n_k} \longrightarrow \lambda \ , \quad \lambda_{n_k}' \longrightarrow \lambda' , \quad \mu_{n_k} \longrightarrow \mu , \quad \mu_{n_k}' \longrightarrow \mu' ,$$
$$\theta_{n_k} \longrightarrow \theta \ , \quad \theta_{n_k}' \longrightarrow \theta' , \quad \phi_{n_k} \longrightarrow \phi \ , \quad \phi_{n_k}' \longrightarrow \phi' ,$$

as $k \longrightarrow \infty$, where $x \in X$ and λ , λ', μ, μ', θ , θ', ϕ , $\phi' \in K$. As in case (a), we can show that

$$0 \neq x \in \liminf G_{n_k} \ ;$$

and so the proof is complete if we can show that $\liminf G_{n_k} \neq X$. Suppose that $\liminf G_{n_k} = X$. Then

$$x_{n_k} \longrightarrow e\ ,\ x'_{n_k} \longrightarrow p(T)e\ ,$$
$$w_{n_k} \longrightarrow \{p(T)\}^2 e\ ,\ w'_{n_k} \longrightarrow \{p(T)\}^3 e\ ,$$

as $k \longrightarrow \infty$. Then, by (13) and (14),

$$\lambda'_{n_k} x_{n_k} - \lambda_{n_k} x'_{n_k} = \begin{vmatrix} \lambda'_{n_k} & \lambda_{n_k} \\ \mu'_{n_k} & \mu_{n_k} \end{vmatrix} v_{n_k} + \lambda'_{n_k} y_{n_k} - \lambda_{n_k} y'_{n_k}\ ;$$

and so

$$\begin{vmatrix} \lambda'_{n_k} & \lambda_{n_k} \\ \mu'_{n_k} & \mu_{n_k} \end{vmatrix} v_{n_k} + \lambda'_{n_k} y_{n_k} - \lambda_{n_k} y'_{n_k} \longrightarrow \lambda' e - \lambda p(T)e \tag{17}$$

as $k \longrightarrow \infty$. Similarly, from (15) and (16),

$$\begin{vmatrix} \theta'_{n_k} & \theta_{n_k} \\ \phi'_{n_k} & \phi_{n_k} \end{vmatrix} v_{n_k} + \theta'_{n_k} z_{n_k} - \theta_{n_k} z'_{n_k} \longrightarrow \theta'\{p(T)\}^2 e - \theta\{p(T)\}^3 e$$

as $k \longrightarrow \infty$. Hence

$$\begin{vmatrix} \lambda'_{n_k} & \lambda_{n_k} \\ \mu'_{n_k} & \mu_{n_k} \end{vmatrix} (\theta'_{n_k} z_{n_k} - \theta_{n_k} z'_{n_k}) - \begin{vmatrix} \theta'_{n_k} & \theta_{n_k} \\ \phi'_{n_k} & \phi_{n_k} \end{vmatrix} (\lambda'_{n_k} y_{n_k} - \lambda_{n_k} y'_{n_k})$$

$$\longrightarrow \begin{vmatrix} \lambda' & \lambda \\ \mu' & \mu \end{vmatrix} (\theta'\{p(T)\}^2 e - \theta\{p(T)\}^3 e) - \begin{vmatrix} \theta' & \theta \\ \phi' & \phi \end{vmatrix} (\lambda' e - \lambda p(T)e)$$

as $k \longrightarrow \infty$. Thus

$$\begin{vmatrix} \lambda' & \lambda \\ \mu' & \mu \end{vmatrix} (\theta'\{p(T)\}^2 e - \theta\{p(T)\}^3 e) - \begin{vmatrix} \theta' & \theta \\ \phi' & \phi \end{vmatrix} (\lambda' e - \lambda p(T)e)$$
$$\in \liminf F_{n_k} = \{0\}\ ,\ \text{by (10)}\ .$$

Hence, by the linear independence of $e, p(T)e, \ldots$,

$$\begin{vmatrix} \lambda' & \lambda \\ \mu' & \mu \end{vmatrix} \theta = \begin{vmatrix} \lambda' & \lambda \\ \mu' & \mu \end{vmatrix} \theta' = 0\ .$$

If $\begin{vmatrix} \lambda' & \lambda \\ \mu' & \mu \end{vmatrix} = 0$, then, by (17),

$$\lambda'_{n_k} y_{n_k} - \lambda_{n_k} y'_{n_k} \longrightarrow \lambda' e - \lambda\, p(T)e$$

as $k \longrightarrow \infty$; and so

$$\lambda' e - \lambda p(T)e \in \liminf F_{n_k} = \{0\} .$$

Therefore, by the linear independence of $e, p(T)e, \dots$,

$$\lambda = \lambda' = 0 .$$

Then, by (13) and (14),

$$\mu'_{n_k} y_{n_k} - \mu_{n_k} y'_{n_k} \longrightarrow \mu' e - \mu\, p(T)e$$

since $x_{n_k} \longrightarrow e$ and $x'_{n_k} \longrightarrow p(T)e$. Hence

$$\mu' e - \mu\, p(T)e \in \liminf F_{n_k} = \{0\} ;$$

and so $\mu = \mu' = 0$. Then, by (13),

$$y_{n_k} \longrightarrow e \in \liminf F_{n_k} = \{0\} ,$$

an obvious contradiction. Hence

$$\begin{vmatrix} \lambda' & \lambda \\ \mu' & \mu \end{vmatrix} \neq 0 , \text{ and } \theta = \theta' = 0 .$$

Then (15) and (16) give

$$\phi'_{n_k} z_{n_k} - \phi_{n_k} z'_{n_k} \longrightarrow \phi' \{p(T)\}^2 e - \phi \{p(T)\}^3 e ,$$

and so

$$\phi' \{p(T)\}^2 e - \phi \{p(T)\}^3 e \in \liminf F_{n_k} = \{0\} .$$

Therefore $\phi = \phi' = 0$, and (15) now gives

$$z_{n_k} \longrightarrow \{p(T)\}^2 e \in \liminf F_{n_k} = \{0\} ,$$

contradicting $\{p(T)\}^2 e \neq 0$.

Thus $\liminf G_{n_k} \neq X$, and the proof is complete.

The following corollary is immediate.

(2.2.3) Corollary. (The Aronszajn-Smith theorem) Let X be a complex (resp. real) normed linear space of dimension greater than 1 (resp. 2), and let T be a compact linear operator on X. Then T has a non-trivial invariant subspace.

Remarks. (1) For complex spaces, Theorem (2.2.2) can be obtained without considering case (b). For, by Theorem (1.2.1), the sequences $\{X_n^i\}$ can be chosen so that $\dim(X_n^i - X_n^{i-1}) = 1$ for $1 \leq i \leq m_n$ and $n \geq 1$. (See [5], Theorem (20.1).)

(2) Corollary (2.2.3) can be proved directly, and in a slightly simpler way as follows. Let e be a cyclic vector for T of norm 1, and construct as above the linear operators T_n on X_n. Then choose sequences $\{X_n^i\}$ of subspaces of X_n, satisfying conditions (i) - (iii) on pp.17,18 (with $j(n) = n$). The proof is then completed as in the proof of Theorem (2.2.2), except that the analysis given to show that $x \in \liminf G_{n_k}$ can be replaced by an application of Lemma (2.1.6) (i). (See [5], Theorem (18.1) for the complex case.)

3. A theorem of J. Feldman.

Using the ideas and methods introduced above, we now obtain the following theorem, which was first proved for Hilbert space by Feldman, [2,11].

(2.3.1) Theorem. Let X be a normed linear space over C (resp. R) of dimension greater than 1 (resp. 2), and let T be a quasi-nilpotent bounded linear operator on X. Suppose, further, that there exists a sequence $\{p_n(T)\}$ of polynomials in T with complex (resp. real) coefficients, and a non-zero compact operator S on X such that $p_n(T) \longrightarrow S$ (in norm) as $n \longrightarrow \infty$. Then T has a non-trivial invariant subspace.

Proof. As in the proof of Theorem (2.2.2), we can assume that X is infinite-dimensional, and write $K = C$ or R according as X is complex or real. We can also assume the existence of a cyclic vector e of norm 1. Further, since $ST = TS$ and S is a non-zero compact operator, we may suppose, by an argument similar to that given on pp.15,16, that $e, Se, S^2e, \ldots$ are linearly independent.

Define X_n, e_n and T_n as in Section 1. From equation (6) on p.17, taking $p(T) = T^r$,

$$d(T^r e_n, X_{n+r-1}) = \prod_{m=n}^{n+r-1} d(Te_m, X_m) \tag{18}$$

for $r \geq 1$, $n \geq 1$. Suppose that $\varliminf_{m \to \infty} d(Te_m, X_m) > 0$. Then there exists a constant $a > 0$ and an integer m_o such that

$$d(Te_m, X_m) \geq a$$

if $m \geq m_o$. From (18),

$$\begin{aligned} \|T^r\| \geq \|T^r e_{m_o}\| &\geq d(T^r e_{m_o}, X_{m_o+r-1}) \\ &= \prod_{m=m_o}^{m_o+r-1} d(Te_m, X_m) \geq a^r \end{aligned}$$

for $r \geq 1$. Hence, for $r \geq 1$,

$$\|T^r\|^{1/r} \geq a > 0,$$

contradicting the quasi-nilpotence of T. Thus

$$\varliminf_{m \to \infty} d(Te_m, X_m) = 0;$$

and so we can find a subsequence $\{j(n)\}$ of $1,2,3,\ldots$ such that

$$d(Te_{j(n)}, X_{j(n)}) \longrightarrow 0 \tag{19}$$

as $n \longrightarrow \infty$.

Choose sequences $\{X_n^i\}_{i=0}^{m_n}$ of subspaces of $X_{j(n)}$ such that

(i) $\{0\} = X_n^0 \subset X_n^1 \subset \dots \subset X_n^{m_n} = X_{j(n)}$ $\qquad (n \geq 1)$,

(ii) $\dim (X_n^i - X_n^{i-1}) \leq 2$ $\qquad (1 \leq i \leq m_n \ , \ n \geq 1)$,

(iii) $T_{j(n)}(X_n^i) \subseteq X_n^i$ $\qquad (0 \leq i \leq m_n \ , \ n \geq 1)$.

Since $e, Se, \dots$ are linearly independent, $Se \neq 0$; and so we can choose α with $0 < \alpha < 1$ and

$$\|Se\| > \alpha \|S\| \tag{20}$$

Following the proof of Theorem (2.2.2), we choose F_n , G_n from the sequence $\{X_n^i\}$ so that $F_n \subset G_n$, $\dim (G_n - F_n) \leq 2$, and

$$d(e, F_n) > \alpha \ , \ d(e, G_n) \leq \alpha \ .$$

Then, for every subsequence $\{n_k\}$, $e \notin \liminf F_{n_k}$; and so $\liminf F_{n_k}$ is a non-trivial invariant subspace for T (by Lemma (2.1.6) (ii)), unless

$$\liminf F_{n_k} = \{0\} \ . \tag{21}$$

We can therefore assume that (21) holds for all subsequences $\{n_k\}$; and we prove that, in this case, $\liminf G_{n_k}$ is non-trivial for some subsequence $\{n_k\}$. This will complete the proof, again by Lemma (2.1.6) (ii) .

There are two cases to be considered.

<u>Case (a)</u>. $\dim (G_n - F_n) = 1$ for infinitely many n. Then, by passing to a subsequence if necessary, we can assume that $\dim (G_n - F_n) = 1$ for all n . Let u_n be a unit vector in G_n orthogonal to F_n , and let x_n , x_n' be nearest points of G_n to e , Se respectively. Since $G_n = [u_n, F_n]$, we can write

$$x_n = \lambda_n u_n + y_n$$

$$x_n' = \lambda_n' u_n + y_n'$$

for $n \geq 1$, where y_n , $y_n' \in F_n$ and λ_n , $\lambda_n' \in K$. As in the proof of Theorem (2.2.2), it is easy to show that $\{|\lambda_n|\}$, $\{|\lambda_n'|\}$ and $\{\|x_n\|\}$ are bounded sequences. Hence, using the compactness of S , we can find a subsequence $\{n_k\}$ such that

$$\lambda_{n_k} \longrightarrow \lambda, \quad \lambda_{n_k}' \longrightarrow \lambda', \quad Sx_{n_k} \longrightarrow x$$

as $k \longrightarrow \infty$, where $\lambda , \lambda' \in K$ and $x \in X$. We prove that

$$x \in \liminf G_{n_k} .$$

Let $\varepsilon > 0$, and let $A > 0$ be such that $\|x_n\| \leq A$ for all n . Choose an integer m such that

$$\|p_m(T) - S\| \leq \varepsilon /4A .$$

Then $\|p_m(T)x_{n_k} - Sx_{n_k}\| \leq \varepsilon /4$ for $k \geq 1$. But $Sx_{n_k} \longrightarrow x$ as $k \longrightarrow \infty$, and so there exists k_1 such that

$$\|p_m(T)x_{n_k} - x\| \leq \varepsilon /2 \tag{22}$$

for $k \geq k_1$. Since $\|x_{n_k}\| \leq A$, by Corollary (2.1.4), there is a constant $M > 0$ such that

$$\|p_m(T)x_{n_k} - p_m(T_{j(n_k)})x_{n_k}\| \leq M\, d(Te_{j(n_k)}, X_{j(n_k)})$$

for $k \geq 1$. Hence, by (19),

$$\|p_m(T)x_{n_k} - p_m(T_{j(n_k)})x_{n_k}\| \longrightarrow 0$$

as $k \longrightarrow \infty$. Therefore, there exists $k_2 \geq k_1$ such that

$$\|p_m(T)x_{n_k} - p_m(T_{j(n_k)})x_{n_k}\| \leq \varepsilon /2 \tag{23}$$

if $k \geq k_2$. Equations (22) and (23) give

$$\|x - p_m(T_{j(n_k)})x_{n_k}\| \leq \varepsilon$$

if $k \geq k_2$. But $p_m(T_{j(n_k)})x_{n_k} \in T_{j(n_k)}G_{n_k} \subseteq G_{n_k}$; and so

$$d(x, G_{n_k}) \leq \|x - p\,(T_{j(n_k)})x_{n_k}\| \leq \varepsilon$$

if $k \geq k_2$. Hence $d(x, G_{n_k}) \longrightarrow 0$ as $k \longrightarrow \infty$; i.e.

$$x \in \liminf G_{n_k} .$$

As in the proof of Theorem (2.2.2), it follows from (20) that $x \neq 0$. Therefore

$$\liminf G_{n_k} \neq \{0\} .$$

The proof in case (a) is completed by showing that, if (21) holds, then

$$\liminf G_{n_k} \neq X .$$

This is done in the same way as in case (a) in the proof of Theorem (2.2.2), using the linear independence of $e, Se, \dots$.

Case (b). $\dim (G_n - F_n) = 2$ for all but a finite number of n . Then, by omitting these, we can assume that $\dim (G_n - F_n) = 2$ for all n . As in case (b) in the proof of Theorem (2.2.2), there exist vectors u_n , v_n in G_n such that

$$\|u_n\| = \|v_n\| = d(u_n, F_n) = d(v_n, [u_n, F_n]) = 1 .$$

Let x_n , x_n' , w_n , w_n' be nearest points of G_n to e , Se , S^2e , S^3e respectively. Since $G_n = [u_n, v_n, F_n]$, we can write

$$\begin{aligned} x_n &= \lambda_n u_n + \mu_n v_n + y_n \\ x_n' &= \lambda_n' u_n + \mu_n' v_n + y_n' \\ w_n &= \theta_n u_n + \phi_n v_n + z_n \\ w_n' &= \theta_n' u_n + \phi_n' v_n + z_n' \end{aligned}$$

where λ_n , λ_n' , μ_n , μ_n' , θ_n , θ_n' , ϕ_n , $\phi_n' \in K$ and y_n , y_n' , z_n , $z_n' \in F_n$

As in the proof of Theorem (2.2.2), we can easily show that the sequences $\{|\lambda_n|\}$, $\{|\lambda_n'|\}$, $\{|\mu_n|\}$, $\{|\mu_n'|\}$, $\{|\theta_n|\}$, $\{|\theta_n'|\}$, $\{|\phi_n|\}$, $\{|\phi_n'|\}$ and $\{\|x_n\|\}$ are bounded. Hence, using the compactness of S , we can find a subsequence $\{n_k\}$ such that

$$Sx_{n_k} \longrightarrow x ,$$
$$\lambda_{n_k} \longrightarrow \lambda , \ \lambda'_{n_k} \longrightarrow \lambda', \ \mu_{n_k} \longrightarrow \mu, \ \mu'_{n_k} \longrightarrow \mu',$$
$$\theta_{n_k} \longrightarrow \theta , \ \theta'_{n_k} \longrightarrow \theta', \ \phi_{n_k} \longrightarrow \phi , \ \phi'_{n_k} \longrightarrow \phi' ,$$

as $k \longrightarrow \infty$, where $x \in X$ and $\lambda, \lambda', \mu, \mu', \theta, \theta', \phi, \phi' \in K$. In the same way as in case (a) above, it can be shown that

$$0 \neq x \in \liminf G_{n_k} .$$

The proof is then completed by showing that (21) implies that

$$\liminf G_{n_k} \neq X .$$

This is done in the same way as in case (b) in the proof of Theorem (2.2.2), using the linear independence of $e, Se, S^2e, \ldots$.

Remarks. (1) As with Theorem (2.2.2), the above theorem can be proved for complex spaces without considering case (b). For, by Theorem (1.2.1), the sequences $\{X_n^i\}$ can be chosen so that $\dim (X_n^i - X_n^{i-1}) = 1$ for all i and n. (See [13].)

(2) In [7], the following generalization of this theorem is proved for complex Hilbert space.

Theorem. Let T be a quasi-nilpotent bounded linear operator on a complex Hilbert space of dimension greater than 1. Suppose that there is a sequence $\{S_n\}$ of compact operators converging weakly to an operator $S \neq 0$, and a sequence $\{p_n\}$ of polynomials such that

$$\|p_n(T) - S_n\| \longrightarrow 0$$

as $n \longrightarrow \infty$. Then T has a non-trivial invariant subspace.

The proof of this theorem relies on essentially Hilbert space methods, and we have been unable to adapt the proof of Theorem (2.3.1) to obtain a corresponding result for normed spaces.

CHAPTER III

SUPER-DIAGONALIZATION OF POLYNOMIALLY COMPACT OPERATORS

From Theorems (1.2.1) and (1.2.4), we see that properties of the spectrum of a linear operator T on a finite-dimensional linear space can be deduced from properties of a nest of invariant subspaces for T. In [25], Ringrose obtains similar results for a compact operator on a complex Banach space which is not necessarily finite-dimensional. He considers nests of invariant subspaces which are maximal in a certain sense, and relates properties of these maximal nests to spectral properties of the operator. The Aronszajn-Smith theorem is used as a principal tool, and we remark that, since this theorem holds without any assumption of completeness, Ringrose's results hold for compact operators on complex normed spaces. Our purpose in this chapter is to obtain corresponding results for polynomially compact operators using the Bernstein-Robinson theorem (Theorem (2.2.2)). Our methods rely heavily on those developed by Ringrose in [25], but we shall deal with both real and complex spaces.

In our discussion we shall make use of the Riesz-Schauder theory of polynomially compact operators. No suitable reference for this could be found in the literature, the nearest results being those given in [36, Chapter 11]. There, an operator T on a complex normed space with the property that T^n is compact for some $n \geq 1$ is discussed. For convenience, we state the corresponding results for a general polynomially compact operator, and remark that they can be obtained by methods similar to those used to develop the Riesz-Schauder theory of compact operators, [8, Chapter 11].

Let T be a bounded linear operator on a normed linear space X over K. Suppose that $p(T)$ is compact, where p is a non-zero polynomial with coefficients in K. We have the following results.

(i) The only possible limit points of $Sp(T)$ are the roots of p.

(ii) If $\lambda \in K$ and $p(\lambda) \neq 0$, then either $\lambda \notin Sp(T)$ or λ is an eigenvalue of T.

(iii) Suppose that $\lambda \in K$ and $p(\lambda) \neq 0$. For $r \geq 0$, let

$$N_r = \{x : (\lambda I - T)^r x = 0\}, \quad R_r = (\lambda I - T)^r X .$$

Then each of the subspaces N_r is finite-dimensional and each R_r is closed. There is a least integer $k \geq 0$ such that $N_k = N_{k+1}$, and k is also the least integer such that $R_k = R_{k+1}$. We call k the <u>index</u> of λ relative to T, and define the <u>algebraic multiplicity</u> of λ relative to T to be the dimension of N_k. Finally, $X = N_k \oplus R_k$, where $\oplus$ denotes a topological direct sum; and $(\lambda I - T)|R_k$ is a homeomorphism.

1. Super-diagonal forms.

Throughout this short section, let X be a normed linear space over K and let $T \in B(X)$.

(3.1.1) <u>Definition</u>. An <u>invariant nest</u> for T is a non-empty set $\mathfrak{F}$ of subspaces of X such that

(i) each $F \in \mathfrak{F}$ is invariant for T,

(ii) $\mathfrak{F}$ is totally ordered by inclusion.

It is clear that the set $\{\{0\}, X\}$ is an invariant nest for T. If $\mathfrak{F}_1, \mathfrak{F}_2$ are invariant nests, let

$$\mathfrak{F}_1 \leq \mathfrak{F}_2$$

mean that each subspace in $\mathcal{J}_1$ belongs to $\mathcal{J}_2$; i.e. $\mathcal{J}_1 \subseteq \mathcal{J}_2$ as sets. Then $\leq$ is a partial ordering on the family of invariant nests. By Zorn's Lemma, it is easy to see that there exist nests which are maximal with respect to $\leq$.

(3.1.2) <u>Definition</u>. A <u>super-diagonal form</u> for T is an invariant nest which is maximal with respect to $\leq$.

We shall study super-diagonal forms for polynomially compact operators in some detail, and show how they exhibit spectral properties of operators. We deal with the real and complex cases in separate sections since the results, though similar, are best stated in slightly different forms.

2. <u>Complex spaces</u>.

Let X be a complex normed space and $T \in B(X)$. Suppose also that $p(T)$ is compact for some non-zero polynomial p .

<u>Notation</u>. Given an invariant nest $\mathcal{J}$ for T and $M \in \mathcal{J}$, put

$$M_- = \mathrm{cl} \bigcup \{L \in \mathcal{J} : L \subset M\} ,$$

where we take $\mathrm{cl}(\emptyset) = \{0\}$. It is clear that M_- is an invariant subspace for T .

(3.2.1) <u>Lemma</u>. Let $\mathcal{J}$ be an invariant nest for T . Then $\mathcal{J}$ is a super-diagonal form if and only if

(i) $\{0\}$ and X belong to $\mathcal{J}$;

(ii) if $\mathcal{G} \subseteq \mathcal{J}$, then

$$\bigcap \{G : G \in \mathcal{G}\} \text{ and } \mathrm{cl} \bigcup \{G : G \in \mathcal{G}\}$$

belong to $\mathcal{J}$;

(iii) $\dim (M - M_-) \leq 1$ for each $M \in \mathcal{J}$.

(Note that (ii) implies that $M_- \in \mathcal{J}$ for each M in $\mathcal{J}$.)

Proof. Suppose that $\mathcal{J}$ is a super-diagonal form.

(i) Since $\mathcal{J} \cup \{ \{0\}, X \}$ is totally ordered by inclusion, the maximality of $\mathcal{J}$ implies that $\{0\}$ and X belong to $\mathcal{J}$.

(ii) Let $\mathcal{G} \subseteq \mathcal{J}$, and let $G_0 = \bigcap \{G : G \in \mathcal{G}\}$. Given $L \in \mathcal{J}$, either there exists $G \in \mathcal{G}$ such that $G \subset L$, or $L \subseteq G$ for all G in $\mathcal{G}$. In the former case, $G_0 \subset L$; and in the latter, $L \subseteq G_0$. Hence $\mathcal{J} \cup \{G_0\}$ is totally ordered by inclusion, and is therefore an invariant nest for T , since $T(G_0) \subseteq G_0$. It now follows from the maximality of $\mathcal{J}$ that $G_0 \in \mathcal{J}$. Similarly, we can show that $\mathrm{cl} \bigcup \{G : G \in \mathcal{G}\}$ belongs to $\mathcal{J}$.

(iii) Let $M \in \mathcal{J}$, and suppose that $\dim (M - M_-) > 1$. The operator $(T|M)_{M_-}$ induced on $M - M_-$ by T is polynomially compact. Therefore, by Theorem (2.2.2), there is a subspace F of $M - M_-$, with $\{0\} \neq F \neq M - M_-$, which is invariant for $(T|M)_{M_-}$. Then, if

$$F_1 = \{x \in X : x + M_- \in F\} ,$$

it is easy to check that F_1 is a (closed) subspace of X which is invariant for T , and that

$$M_- \subset F_1 \subset M .$$

From the definition of M_- , $F_1 \notin \mathcal{J}$. Also, if $G \in \mathcal{J}$, then either $G \subseteq M_-$, in which case $G \subset F_1$, or $M \subseteq G$ and $F_1 \subset G$. Thus $\mathcal{J} \cup \{F_1\}$ is totally ordered and is an invariant nest for T . But this contradicts the maximality of $\mathcal{J}$ since $F_1 \notin \mathcal{J}$. Hence $\dim (M - M_-) \leq 1$.

Conversely, suppose that $\mathcal{J}$ satisfies (i) - (iii) and is not a super-diagonal form. Then there is a subspace F of X such that (a) F is invariant for T , (b) $\mathcal{J} \cup \{F\}$ is totally ordered by inclusion, and (c) $F \notin \mathfrak{J}$. Let $M = \mathrm{cl} \bigcup \{L \in \mathfrak{J} : L \subset F\}$ and $N = \bigcap \{L \in \mathfrak{J} : F \subset L\}$. By (ii), M and N belong to $\mathfrak{J}$; and from (b) it is easy to show that $N_- = M$. But $F \notin \mathfrak{J}$, and so

$$M \subset F \subset N .$$

Thus $\dim (M - M_-) \geq 2$, contradicting (iii). Hence $\mathfrak{J}$ is a super-diagonal form.

Throughout the rest of this section let $\mathfrak{J}$ be a fixed super-diagonal form for T.

<u>Notation</u>. Let $\mathfrak{J}_1 = \{M \in \mathfrak{J} : M \neq M_-\}$. If $M \in \mathfrak{J}_1$ then $\dim (M - M_-) = 1$. Thus, if $z_M \in M \setminus M_-$, we can write

$$Tz_M = \alpha_M z_M + y_M ,$$

where $\alpha_M \in C$ and $y_M \in M_-$. Further, α_M depends on M but not on the particular z_M chosen. (In fact α_M is the unique eigenvalue of the operator $(T|M)_{M_-}$ on $M - M_-$.)

(3.2.2) <u>Lemma</u>. Let $\alpha \in \mathrm{Sp}(T)$, and suppose that $p(\alpha) \neq 0$. Let x_0 be any eigenvector corresponding to α, and set

$$M = \bigcap \{L \in \mathfrak{J} : x_0 \in L\} .$$

Then $M \in \mathfrak{J}_1$ and $\alpha_M = \alpha$.

<u>Proof</u>. By Lemma (3.2.1), $M \in \mathfrak{J}$. Let $\mathfrak{G} = \{L \in \mathfrak{J} : L \subset M\}$. Since $0 \neq x_0 \in M$, $\{0\} \in \mathfrak{G}$. Therefore $\mathfrak{G} \neq \emptyset$. Given $L \in \mathfrak{G}$, choose z_L in L such that

$$\|x_0 - z_L\| \leq 2\, d(x_0, L) .$$

Since $Tx_0 = \alpha x_0$ and $T(L) \subseteq L$,

$$p(T)(x_0 - z_L) + L = p(\alpha)(x_0 - z_L) + L ,$$

and hence

$$\begin{aligned}\|p(T)(x_o - z_L) + L\| &= |p(\alpha)|\|(x_o - z_L) + L\| \\ &= |p(\alpha)|\ d(x_o, L) \\ &\geq \tfrac{1}{2}|p(\alpha)|\|x_o - z_L\| .\end{aligned}$$

From the definition of M, $x_o \notin L$; and so $\|x_o - z_L\| > 0$. Also, $x_o - z_L \in M$. Therefore, if

$$S_L = \{x \in M : \|x\| = 1 \text{ and } \|p(T)x + L\| \geq \tfrac{1}{2}|p(\alpha)|\} ,$$

then $S_L \neq \emptyset$ for each L in $\mathcal{G}$. Since $\mathcal{G}$ is totally ordered, it is easily seen that $\{S_L : L \in \mathcal{G}\}$ is totally ordered also; and hence so is $\{cl(p(T)S_L) : L \in \mathcal{G}\}$. From the compactness of $p(T)$, it follows that there exists y in $\bigcap_{L \in \mathcal{G}} cl(p(T)S_L)$. Then $y \in M$ and $\|y + L\| \geq \frac{1}{2}|p(\alpha)|$ for all L in $\mathcal{G}$. Therefore $y \notin M_-$ and $M \neq M_-$. Hence $M \in \mathcal{J}_1$.

If $x_o \in M_-$ then, by the definition of M, $M \subseteq M_-$ and so $M = M_-$, contradicting what we have just shown. Therefore $x_o \in M \setminus M_-$. Also, $Tx_o = \alpha x_o$, and hence $\alpha_M = \alpha$.

The following lemma gives us a result in the opposite direction. Its proof borrows an idea from the proof of Lemma (5.1) in [35].

(3.2.3) <u>Lemma</u>. Let $M \in \mathcal{J}_1$. Then $\alpha_M \in Sp(T)$.

<u>Proof</u>. Suppose that $\alpha_M \notin Sp(T)$; i.e. there exists $S \in B(X)$ such that $S(\alpha_M I - T) = (\alpha_M I - T)S = I$. Let $\bar{X}$ denote the completion of X and $\bar{S}$, $\bar{T}$ the unique continuous extensions of S, T respectively to $\bar{X}$. $p(\bar{T})$ is compact on $\bar{X}$, and hence the only possible limit points of $Sp(\bar{T})$ are roots of the polynomial p. Therefore $C \setminus Sp(\bar{T})$ is connected.

Let $\bar{M}$ be the closure of M in $\bar{X}$. If $|\lambda| > \|\bar{T}\|$,

$$(\lambda \bar{I} - \bar{T})^{-1} = \sum_{n=0}^{\infty} \lambda^{-n-1}\, \bar{T}^n ;$$

and hence $(\lambda \bar{I} - \bar{T})^{-1}\, \bar{M} \subseteq \bar{M}$, since $\bar{T}(\bar{M}) \subseteq \bar{M}$. It follows that, if f is a continuous linear functional on $\bar{X}$ such that $f(\bar{M}) = \{0\}$, and if

$x \in \bar{M}$, then

$$((\lambda \bar{I} - \bar{T})^{-1}x, f) = 0$$

for $|\lambda| > \|\bar{T}\|$. But the function $\lambda \longrightarrow ((\lambda \bar{I} - \bar{T})^{-1}x, f)$ is analytic on $C \setminus Sp(\bar{T})$, which is connected. Therefore

$$((\lambda \bar{I} - \bar{T})^{-1}x, f) = 0$$

if $x \in \bar{M}$, $f \in \bar{M}^{\perp}$, and $\lambda \in C \setminus Sp(\bar{T})$. Hence

$$(\lambda \bar{I} - \bar{T})^{-1}x \in \bar{M}$$

if $x \in \bar{M}$ and $\lambda \in C \setminus Sp(\bar{T})$; i.e.

$$(\lambda \bar{I} - \bar{T})^{-1}(\bar{M}) \subseteq \bar{M}$$

if $\lambda \in C \setminus Sp(\bar{T})$. Now $\bar{S}(\alpha_M \bar{I} - \bar{T}) = (\alpha_M \bar{I} - \bar{T})\bar{S} = \bar{I}$. Therefore $\alpha_M \in C \setminus Sp(\bar{T})$ and $(\alpha_M \bar{I} - \bar{T})^{-1} = \bar{S}$. Hence $\bar{S}(\bar{M}) \subseteq \bar{M}$, from which it follows that

$$S(M) \subseteq \bar{M} \cap X = M .$$

Also, from the definition of α_M ,

$$(\alpha_M I - T)(M) \subseteq M_- .$$

Hence, for $x \in M$,

$$\begin{aligned} x = (\alpha_M I - T)Sx &\in (\alpha_M I - T)S(M) \\ &\subseteq (\alpha_M I - T)(M) \subseteq M_- . \end{aligned}$$

This gives $M = M_-$, contradicting $M \in \mathfrak{F}_1$. Therefore $\alpha_M \in Sp(T)$.

Remark. If $p(\alpha_M) \neq 0$ this result could be obtained more easily. For, in this case, the operator $T|M$ on M is polynomially compact and, by the Riesz-Schauder theory, has α_M as an eigenvalue.

(3.2.4) Definition. Given $\alpha \in C$, define the diagonal multiplicity of α in $\mathfrak{F}$ to be the number of distinct M's in $\mathfrak{F}_1$ for which $\alpha_M = \alpha$. (The diagonal multiplicity is thus either a non-negative integer or $+\infty$.)

(3.2.5) <u>Lemma</u>. Let $\alpha \in C$ with $p(\alpha) \neq 0$. Then the diagonal multiplicity of α in $\mathcal{J}$ is equal to the algebraic multiplicity of α relative to T (and in particular is finite).

<u>Proof</u>. If $\alpha \notin Sp(T)$ then the algebraic multiplicity of α is 0 . Also, by Lemma (3.2.3), the diagonal multiplicity of α is 0 . Hence, we can assume that $\alpha \in Sp(T)$.

Firstly, we suppose that α has index 1 relative to T . Let $\mathcal{G} = \{M \in \mathcal{J}_1 : \alpha_M = \alpha\}$ and $N = \{x \in X : Tx = \alpha x\}$. For each M in $\mathcal{G}$, the operator $T|M$ is polynomially compact and, since $(\alpha I - T)(M) \subseteq M_- \subset M$, $\alpha \in Sp(T|M)$. Further, α has index 1 relative to $T|M$. Hence, by the Riesz-Schauder theory,

$$M = (\alpha I - T)(M) \oplus N \cap M .$$

But $(\alpha I - T)(M) \subseteq M_-$ and $M \neq M_-$. Therefore $N \cap M \not\subseteq M_-$. For each M in $\mathcal{G}$, let $x_M \in N \cap M \setminus M_-$. We show that $\{x_M : M \in \mathcal{G}\}$ is a linearly independent set. Suppose not. Then, since each $x_M \neq 0$, we can find distinct subspaces $M_1, M_2, \ldots, M_n$ in $\mathcal{G}$ (with $n \geq 2$) such that

$$\lambda_1 x_{M_1} + \ldots + \lambda_n x_{M_n} = 0 ,$$

where $0 \neq \lambda_i \in C$ for $1 \leq i \leq n$. Also, by suitably renumbering if necessary, we can assume that $M_1 \subset M_2 \subset \ldots \subset M_n$. Then

$$x_{M_n} \in M_{n-1} \subseteq (M_n)_- ,$$

contradicting $x_{M_n} \notin (M_n)_-$. Therefore $\{x_M : M \in \mathcal{G}\}$ is a linearly independent subset of N . It follows that the diagonal multiplicity of α is less than or equal to the dimension of N , which equals the algebraic multiplicity of α . Also, N is finite-dimensional. Hence the diagonal multiplicity of α is finite and equals d , say.

By Lemma (3.2.2), $\mathcal{G} \neq \emptyset$; i.e. $d \geq 1$. Thus we can write

$$\mathcal{G} = \{M_1, \ldots, M_d\} ,$$

where $M_1 \subset M_2 \subset \ldots \subset M_d$. Also, for convenience we write

$$x_i = x_{M_i} \qquad (1 \leq i \leq d) .$$

We prove, by induction on i , that

$$M_i \cap N = [x_1, \ldots, x_i] \qquad (1 \leq i \leq d) \qquad (1)$$

Suppose that $x \in M_1 \cap N$. We can write

$$x = \lambda x_1 + y ,$$

where $y \in (M_1)_-$ and $\lambda \in C$. Since x and x_1 belong to N , it follows that $y \in N$. If $y \neq 0$ then, by Lemma (3.2.2),

$$\bigcap \{L \in \mathcal{J} : y \in L\} \in \mathcal{G} ;$$

and hence $\bigcap \{L \in \mathcal{J} : y \in L\} = M_j$ for some j . Since $y \in (M_1)_- \in \mathcal{J}$, it follows that

$$M_j \subseteq (M_1)_- \subset M_1 .$$

But this contradicts $M_1 \subset \ldots \subset M_d$. Therefore $y = 0$ and $x = \lambda x_1$. Hence (1) holds for $i = 1$.

Suppose now that $1 < i \leq d$ and that

$$M_{i-1} \cap N = [x_1, \ldots, x_{i-1}] .$$

Let $x = \lambda x_i + y \in M_i \cap N$, where $y \in (M_i)_-$ and $\lambda \in C$. Then, since x , $x_i \in N$, $y \in N$. If $y = 0$, $x = \lambda x_i \in [x_1, .., x_i]$. If $y \neq 0$, it follows as above from Lemma (3.2.2) that

$$y \in \bigcap \{L \in \mathcal{J} : y \in L\} = M_j$$

for some j . Since $y \in (M_i)_-$, it follows that

$$M_j \subseteq (M_i)_- \subset M_i .$$

Therefore $j \leq i-1$; and so, by the induction hypothesis,

$$y \in M_j \subseteq M_{i-1} = [x_1, \ldots, x_{i-1}] .$$

Therefore $x \in [x_1,\ldots,x_i]$. We have thus shown that

$$M_i \cap N \subseteq [x_1,\ldots,x_i] .$$

Conversely, it is clear that $[x_1,\ldots,x_i] \subseteq M_i \cap N$. Hence

$$M_i \cap N = [x_1,\ldots,x_i] ,$$

and (1) is established by induction. In particular,

$$M_d \cap N = [x_1,\ldots,x_d] . \tag{2}$$

Now, if $0 \neq y \in N$ then, by Lemma (3.2.2), $y \in M_i$ for some i . Therefore $N \subseteq M_d$. It follows from (2) that

$$N = [x_1,\ldots,x_d] .$$

Further, we have already seen that $x_1,\ldots,x_d$ are linearly independent, and so $d = \dim N$. This completes the proof in the case when the index of α is 1 .

Suppose now that the index of α is k $(k > 1)$. Let s be the polynomial

$$s(z) = (z - \alpha)^k - (-\alpha)^k .$$

Suppose that

$$p(z) = A(z - \alpha_1)(z - \alpha_2)\ldots(z - \alpha_r) ,$$

where $A \neq 0$. Define the polynomial q by

$$q(z) = A\{z - s(\alpha_1)\}\{z - s(\alpha_2)\}\ldots\{z - s(\alpha_r)\} .$$

Then

$$q(s(z)) = A\{s(z) - s(\alpha_1)\} \ldots \{s(z) - s(\alpha_r)\} = p(z)r(z) ,$$

where r is some polynomial .

Let $S = s(T) \in B(X)$. Then

$$q(S) = q(s(T)) = p(T)r(T) ;$$

and so $q(S)$ is compact. Also, $\mathfrak{I}$ is a super-diagonal form for S . Let $M \in \mathfrak{I}_1$, and suppose that $x_M \in M \setminus M_-$. We have

$$Tx_M - \alpha_M x_M \in M_- ;$$

and so

$$Sx_M - s(\alpha_M)x_M = s(T)x_M - s(\alpha_M)x_M \in M_- .$$

Now $\alpha_M = \alpha$ if and only if $s(\alpha_M) = -(-\alpha)^k$. Hence the diagonal multiplicity of α in $\mathfrak{F}$ w.r.t. T equals the diagonal multiplicity of $-(-\alpha)^k$ in $\mathfrak{F}$ w.r.t. S . Since k is the index of α relative to T , it easily follows that the index of $-(-\alpha)^k$ relative to S is 1 . Furthermore, q(S) is compact and

$$\begin{aligned} q(-(-\alpha)^k) &= A\{-(-\alpha)^k - s(\alpha_1)\}\ldots\{-(-\alpha)^k - s(\alpha_r)\} \\ &= (-)^r A\,(\alpha_1 - \alpha)^k \ldots (\alpha_r - \alpha)^k \\ &\neq 0 , \end{aligned}$$

since $p(\alpha) \neq 0$ and $\alpha_1, \ldots, \alpha_r$ are the roots of p . It follows from what has been proved above that the algebraic multiplicity of $-(-\alpha)^k$ relative to S is equal to the diagonal multiplicity of $-(-\alpha)^k$ in $\mathfrak{F}$ w.r.t. S , and hence equals the diagonal multiplicity of α in $\mathfrak{F}$ w.r.t. T . It is easy to verify that the algebraic multiplicity of $-(-\alpha)^k$ relative to S equals the algebraic multiplicity of α relative to T . The required result now follows immediately .

We summarise the lemmas given above in the following theorem, which corresponds to Theorem 2 in [25] .

(3.2.6) Theorem. Let X be a complex normed space and $T \in B(X)$. Suppose that p is a non-zero polynomial such that p(T) is compact. Let $\mathfrak{F}$ be a super-diagonal form for T ; and define $\mathfrak{F}_1$ and the complex numbers α_M $(M \in \mathfrak{F}_1)$ as above . Then

(i) $Sp(T) \cap \{\alpha \in C : p(\alpha) \neq 0\} \subseteq \{\alpha_M : M \in \mathfrak{F}_1\}$;

(ii) $\{\alpha_M : M \in \mathcal{F}_1\} \subseteq Sp(T)$;

(iii) if $p(\alpha) \neq 0$, then the diagonal multiplicity of α in $\mathcal{F}$ equals the algebraic multiplicity of α relative to T .

The following corollary corresponds to the Corollary to Theorem 2 in [25] .

(3.2.7) Corollary. If $\mathcal{F}_1 = \emptyset$, then $Sp(T) \subseteq \{\alpha \in C : p(\alpha) = 0\}$.

Proof. Immediate from (i) .

Remark. The results presented in this section have been obtained by Ringrose in the case when T is compact (i.e. when $p(z) = z$). For the most part, our proofs are simple adaptations of the proofs given by Ringrose of the corresponding results in the compact case. This is possible because the Bernstein-Robinson theorem enables us to characterize the super-diagonal forms for a polynomially compact operator (Lemma (3.2.1)) in the same way that the Aronszajn-Smith theorem enabled Ringrose to characterize them for a compact operator.

3. Real spaces.

Throughout this section let X be a real normed space and let $T \in B(X)$. Also, suppose that p is a non-zero polynomial with real coefficients such that $p(T)$ is compact. As in the above section, if $\mathcal{F}$ is an invariant nest for T and $M \in \mathcal{F}$, we set

$$M_- = cl \bigcup \{L \in \mathcal{F} : L \subset M\} ,$$

where we take $cl(\emptyset) = \{0\}$. Corresponding to Lemma (3.2.1) in the complex case, the following lemma characterizes the super-diagonal forms in the real case.

(3.3.1) Lemma. Let $\mathcal{J}$ be an invariant nest for T. Then $\mathcal{J}$ is a super-diagonal form if and only if

(i) $\{0\}$ and X belong to $\mathcal{J}$;

(ii) if $\mathcal{G} \subseteq \mathcal{J}$, then

$$\bigcap\{G : G \in \mathcal{G}\} \text{ and } \mathrm{cl}\bigcup\{G : G \in \mathcal{G}\}$$

belong to $\mathcal{J}$;

(iii) $\dim(M - M_-) \leq 2$ for each M in $\mathcal{J}$; and if $\dim(M - M_-) = 2$, then the operator $(T|M)_{M_-}$ induced by T on $M - M_-$ is irreducible.

Proof. Suppose that $\mathcal{J}$ is a super-diagonal form; i.e. $\mathcal{J}$ is maximal w.r.t. the partial ordering $\leq$ on the invariant nests for T introduced in Section 1. (i) and (ii) follow from the maximality of $\mathcal{J}$ in exactly the same way as in the proof of Lemma (3.2.1). For (iii), we note firstly that, if F is a non-trivial subspace of $M - M_-$ which is invariant for $(T|M)_{M_-}$ and if

$$F_1 = \{x \in X : x + M_- \in F\},$$

then F_1 is invariant for T and $M_- \subset F_1 \subset M$. It follows easily that $F_1 \notin \mathcal{J}$ and that $\mathcal{J} \cup \{F_1\}$ is totally ordered by inclusion. This contradicts the maximality of $\mathcal{J}$. Therefore $(T|M)_{M_-}$ has no non-trivial invariant subspaces. Thus, by Theorem (2.2.2), $\dim(M - M_-) \leq 2$; and also, if $\dim(M - M_-) = 2$ then $(T|M)_{M_-}$ is irreducible.

Conversely, suppose that $\mathcal{J}$ satisfies (i) - (iii) and is not a super-diagonal form. Then there exists a subspace F of X such that (a) F is invariant for T, (b) $\mathcal{J} \cup \{F\}$ is totally ordered by inclusion, and (c) $F \notin \mathcal{J}$. Let $M = \mathrm{cl}\bigcup\{L \in \mathcal{J} : L \subset F\}$ and $N = \bigcap\{L \in \mathcal{J} : F \subset L\}$. By (ii), M and N belong to $\mathcal{J}$. As in the proof of Lemma (3.2.1), it is easily seen that $N_- = M$ and that $M \subset F \subset N$. Therefore $\dim (N - N_-) \geq 2$, and so, by (iii) , $\dim (N - N_-) = 2$. But $\{x + N_- : x \in F\}$ is a non-trivial invariant subspace for $(T|N)_{N_-}$, and this contradicts (iii) . Hence $\mathcal{J}$ is a super-diagonal form for T .

For the rest of this section let $\mathcal{J}$ be a fixed super-diagonal form for T .

Notation. Let

$$\mathcal{J}_1 = \{M \in \mathcal{J} : \dim (M - M_-) = 1\}$$

and

$$\mathcal{J}_2 = \{M \in \mathcal{J} : \dim (M - M_-) = 2\} .$$

If $M \in \mathcal{J}_1$ and $z_M \in M \setminus M_-$, we can write

$$Tz_M = \beta_M z_M + y_M ,$$

where $\beta_M \in R$ and $y_M \in M_-$. Further, β_M is the unique eigenvalue of the operator $(T|M)_{M_-}$ on $M - M_-$, and hence depends on M but not on the particular z_M chosen.

If $M \in \mathcal{J}_2$ then the operator $(T|M)_{M_-}$ on $M - M_-$ is irreducible. Therefore, by Lemma (1.2.3), there exist y_M and z_M in $M \setminus M_-$ such that, for some α_M , β_M in R with $\beta_M \neq 0$,

$$Ty_M = \alpha_M y_M - \beta_M z_M + u_M$$
$$Tz_M = \beta_M y_M + \alpha_M z_M + v_M ,$$

where $u_M , v_M \in M_-$. Let

$$D_M = \{\alpha_M + i\beta_M , \alpha_M - i\beta_M \} .$$

Then D_M is the set of eigenvalues of the operator $(T|M)_{M_-}$, and hence depends on M but not on the particular y_M and z_M chosen.

(3.3.2) Definitions. (i) Given $\beta \in R$, define the diagonal multi-

<u>plicity</u> of β in $\mathfrak{J}$ to be the number (either finite or $+\infty$) of distinct M's in $\mathfrak{J}_1$ such that $\beta = \beta_M$.

(ii) Given $\gamma \in C \setminus R$, define the <u>diagonal multiplicity</u> of γ in $\mathfrak{J}$ to be the number (either finite or $+\infty$) of distinct M's in $\mathfrak{J}_2$ such that $\gamma \in D_M$.

As in Chapter I, we denote by X_c the complexification of X and by T_c the linear operator induced on X_c by T. Further, the spectrum of T is defined by

$$Sp(T) = Sp(T_c) .$$

(3.3.3) <u>Definition</u>. Suppose that $\gamma \in C \setminus R$ and $p(\gamma) \neq 0$. Define the <u>algebraic multiplicity</u> of γ relative to T to be the algebraic multiplicity of γ relative to T_c.

<u>Remark</u>. Suppose that $\gamma \in R$ and that $p(\gamma) \neq 0$. It is easy to show that the algebraic multiplicities of γ relative to T and T_c are equal. Hence, even for real γ Definition (3.3.3) is consistent with the definition of algebraic multiplicity given on p.31.

We can now state the theorem which corresponds in the real case to Theorem (3.2.6).

(3.3.4) <u>Theorem</u>. Let X be a real normed space and let $T \in B(X)$. Suppose that p is a polynomial with real coefficients such that $p(T)$ is compact. Also, let $\mathfrak{J}$ be a super-diagonal form for T and, with the above notation, let

$$D = \{\beta_M : M \in \mathfrak{J}_1\} \cup \bigcup \{D_M : M \in \mathfrak{J}_2\} .$$

Then

(i) $Sp(T) \cap \{\gamma \in C : p(\gamma) \neq 0\} \subseteq D$;

(ii) $D \subseteq Sp(T)$;

(iii) if $\gamma \in C$ and $p(\gamma) \neq 0$, then the diagonal multiplicity of γ in $\mathcal{J}$ equals the algebraic multiplicity of γ relative to T .

Proof. The method of proof is as follows. We construct from $\mathcal{J}$ a super-diagonal form $\mathcal{J}_c$ for T_c . Since p has real coefficients, $\{p(T)\}_c = p(T_c)$. Hence $p(T_c)$ is compact. The required results for T and $\mathcal{J}$ follow from an application of Theorem (3.2.6) to T_c and $\mathcal{J}_c$.

Given a subspace M of X , let

$$M_c = \{x + iy : x , y \in M\} .$$

Then M_c is a (closed) subspace of X_c . Also, the following results are easily verified.

(A) If $T(M) \subseteq M$ then $T_c(M_c) \subseteq M_c$.

(B) If $\mathcal{G}$ is a non-empty collection of subspaces of X , then

$$\{\bigcap \mathcal{G}\}_c = \bigcap\{G_c : G \in \mathcal{G}\} .$$

(C) If $\mathcal{G}$ is a non-empty nest of subspaces of X , then

$$\{cl(\bigcup \mathcal{G})\}_c = cl \bigcup \{G_c : G \in \mathcal{G}\} .$$

Given M in $\mathcal{J}_2$,

$$\dim_R(M - M_-) = \dim_C(M_c - (M_-)_c) = 2 ,$$

where $\dim_K$ denotes the dimension over K (= R,C). By (A), M_c and $(M_-)_c$ are invariant for T_c . Therefore, for each M in $\mathcal{J}_2$ we can choose a subspace F_M of X_c such that $T_c(F_M) \subseteq F_M$ and $(M_-)_c \subset F_M \subset M_c$. Let

$$\mathcal{J}_c = \{M_c : M \in \mathcal{J}\} \cup \{F_M : M \in \mathcal{J}_2\} .$$

We shall prove that $\mathcal{J}_c$ is a super-diagonal form for T_c .

Since $\mathcal{J}$ is totally ordered by inclusion, so is $\{M_c : M \in \mathcal{J}\}$.

If $L, M \in \mathfrak{J}$ and $L \neq M$, then either $L \subset M$ or $M \subset L$. In the former case $L \subseteq M_-$, and so

$$F_L \subset L_c \subseteq (M_-)_c \subset F_M .$$

Similarly, in the latter case $F_M \subset F_L$. Therefore $\{F_M : M \in \mathfrak{J}_2\}$ is totally ordered by inclusion. Suppose that $M \in \mathfrak{J}$ and $L \in \mathfrak{J}_2$. Either $M \subset L$ or $L \subseteq M$. In the former case, $M \subseteq L_-$ and hence

$$M_c \subseteq (L_-)_c \subset F_L ;$$

and in the latter case

$$F_L \subset L_c \subseteq M_c .$$

It now follows that $\mathfrak{J}_c$ is totally ordered by inclusion, and hence is an invariant nest for T. Suppose that $\mathfrak{J}_c$ is not a super-diagonal form. Then there is a subspace F_o of X_c such that (a) F_o is invariant for T_c, (b) $\mathfrak{J}_c \cup \{F_o\}$ is totally ordered by inclusion, and (c) $F_o \notin \mathfrak{J}_c$. Let

$$G_o = \bigcap \{G \in \mathfrak{J} : F_o \subset G_c\} .$$

By Lemma (3.3.1) (ii), $G_o \in \mathfrak{J}$. By result (B) given above,

$$(G_o)_c = \bigcap \{G_c : G \in \mathfrak{J} \text{ and } F_o \subset G_c\} ,$$

and so $F_o \subseteq (G_o)_c$. But $(G_o)_c \in \mathfrak{J}_c$ and $F_o \notin \mathfrak{J}_c$. Therefore

$$F_o \subset (G_o)_c . \tag{3}$$

Given L in $\mathfrak{J}$, $L \subset G_o$ if and only if $L_c \subset F_o$; and hence

$$(G_o)_- = \mathrm{cl} \bigcup \{ L \in \mathfrak{J} : L_c \subset F_o\} .$$

Then, by (C), $\{(G_o)_-\}_c = \mathrm{cl} \bigcup \{L_c : L \in \mathfrak{J}$ and $L_c \subset F_o\}$; from which it follows that

$$\{(G_o)_-\}_c \subset F_o . \tag{4}$$

From (3) and (4) we see that

$$\dim_C \{(G_o)_c - \{(G_o)_-\}_c\} \geq 2 .$$

But

$$\dim_C\{(G_0)_c - \{(G_0)_-\}_c\} = \dim_R\{G_0 - (G_0)_-\} \leq 2 \tag{5}$$

by Lemma (3.3.1) (iii). Hence $\dim_R\{G_0 - (G_0)_-\} = 2$ and $G_0 \in \mathfrak{J}_2$.
Then either $F_0 \subset F_{G_0}$ or $F_{G_0} \subset F_0$. In the former case, using (3) and (4), we get

$$\{(G_0)_-\}_c \subset F_0 \subset F_{G_0} \subset (G_0)_c;$$

and in the latter

$$\{(G_0)_-\}_c \subset F_{G_0} \subset F_0 \subset (G_0)_c \quad .$$

Thus, in both cases

$$\dim_C\{(G_0)_c - \{(G_0)_-\}_c\} \geq 3 ,$$

contradicting (5). Hence $\mathfrak{J}_c$ is a super-diagonal form for T_c .

We now determine $(\mathfrak{J}_c)_1$, i.e. we determine those subspaces G in $\mathfrak{J}_c$ for which $G_- \neq G$. To do this, note firstly that if $G_1, G_2 \in \mathfrak{J}_c$, $G_1 \subset G_2$, and $\dim_C(G_2 - G_1) = 1$, then $G_1 = (G_2)_-$. Now, if $M \in \mathfrak{J}_2$ then

$$\dim_C\{M_c - F_M\} = \dim_C\{F_M - (M_-)_c\} = 1 .$$

Hence

$$(M_c)_- = F_M \ , \ (F_M)_- = (M_-)_c \ ;$$

and so M_c , $F_M \in (\mathfrak{J}_c)_1$. Also, if $M \in \mathfrak{J}_1$ then

$$\dim_C\{M_c - (M_-)_c\} = \dim_R\{M - M_-\} = 1 .$$

Hence $(M_c)_- = (M_-)_c$, and $M_c \in (\mathfrak{J}_c)_1$. We have thus proved that

$$\{M_c : M \in \mathfrak{J}_1 \cup \mathfrak{J}_2\} \cup \{F_M : M \in \mathfrak{J}_2\} \subseteq (\mathfrak{J}_c)_1 \ . \tag{6}$$

If $M \in \mathfrak{J} \setminus \{\mathfrak{J}_1 \cup \mathfrak{J}_2\}$, then $M = M_- = \mathrm{cl} \bigcup \{L \in \mathfrak{J} : L \subset M\}$.

Hence, by (C),

$$M_c = \mathrm{cl} \bigcup \{L_c : L \in \mathfrak{J} \text{ and } L \subset M\} .$$

But $L \subset M$ if and only if $L_c \subset M_c$. Therefore

$$M_c = \mathrm{cl} \bigcup \{L_c : L \in \mathfrak{J} \text{ and } L_c \subset M_c \}$$

$$\subseteq \operatorname{cl} \bigcup \{G : G \in \mathfrak{J}_c \text{ and } G \subset M_c\}$$
$$= (M_c)_- .$$

Hence $M_c \notin (\mathfrak{J}_c)_1$ if $M \in \mathfrak{J} \setminus \{\mathfrak{J}_1 \cup \mathfrak{J}_2\}$. Equation (6) now gives

$$\{M_c : M \in \mathfrak{J}_1 \cup \mathfrak{J}_2\} \cup \{F_M : M \in \mathfrak{J}_2\} = (\mathfrak{J}_c)_1 .$$

With the notation of Section 2 for the super-diagonal form $\mathfrak{J}_c$, it is clear that $\alpha_{M_c} = \beta_M$ for each M in $\mathfrak{J}_1$. If $M \in \mathfrak{J}_2$, by considering the operator $(T_c | M_c)_{(M_-)_c}$ on $M_c - (M_-)_c$, it is easy to see that

$$\{\alpha_M , \alpha_{F_M}\} = \operatorname{Sp}((T|M)_{M_-}) = D_M .$$

Thus

$$\{\alpha_G : G \in (\mathfrak{J}_c)_1\} = \{\beta_M : M \in \mathfrak{J}_1\} \cup \bigcup \{D_M : M \in \mathfrak{J}_2\} = D . \quad (7)$$

As remarked above, the operator $p(T_c)$ is compact. Noting that $\operatorname{Sp}(T) = \operatorname{Sp}(T_c)$ by definition, results (i) and (ii) follow directly from (7), together with Theorem (3.2.6) (i),(ii) applied to the super-diagonal form $\mathfrak{J}_c$ for T_c .

To prove (iii), we show firstly that, for $\gamma \in C$, the diagonal multiplicity of γ in $\mathfrak{J}$ equals the diagonal multiplicity of γ in $\mathfrak{J}_c$. Put $d(\gamma)$ = the diagonal multiplicity of γ in $\mathfrak{J}_c$. Then

$d(\gamma)$ = the number of distinct G's in $(\mathfrak{J}_c)_1$ with $\alpha_G = \gamma$.

If $M \in \mathfrak{J}_2$, then $\{\alpha_M, \alpha_{F_M}\} = D_M \subseteq C \setminus R$. Therefore, for $\gamma \in R$

$d(\gamma)$ = the number of distinct M_c's with $M \in \mathfrak{J}_1$ and $\alpha_{M_c} = \gamma$.

But $L = M$ if and only if $L_c = M_c$. Hence

$d(\gamma)$ = the number of distinct M's in $\mathfrak{J}_1$ with $\beta_M = \gamma$,

i.e.

$d(\gamma)$ = the diagonal multiplicity of γ in $\mathfrak{J}$.

If $\gamma \in C \setminus R$, since $\alpha_{M_c} \in R$ for M in $\mathfrak{J}_1$, we have

$d(\gamma)$ = the number of distinct G's in $\{M_c : M \in \mathcal{J}_2\} \cup \{F_M : M \in \mathcal{J}_2\}$ with $\alpha_G = \gamma$.

If $M \in \mathcal{J}_2$ then $\{\alpha_M, \alpha_{F_M}\} = D_M$ consists of a pair of non-real complex numbers which are the complex conjugates of each other. Thus $\alpha_M \neq \alpha_{F_M}$. Also. if $L, M \in \mathcal{J}_2$ and $L \neq M$, then L_c, M_c, F_L, F_M are distinct subspaces. Hence

$d(\gamma)$ = the number of distinct M's in $\mathcal{J}_2$ with $\gamma \in D_M$,

i.e., by Definition (3.3.2) (ii),

$d(\gamma)$ = the diagonal multiplicity of γ in $\mathcal{J}$.

We have thus shown that, for all γ in C, the diagonal multiplicities of γ in $\mathcal{J}$ and $\mathcal{J}_c$ are equal.

It now follows from Theorem (3.2.6) (iii) that, if $\gamma \in C$ and $p(\gamma) \neq 0$, then the diagonal multiplicity of γ in $\mathcal{J}$ equals the algebraic multiplicity of γ relative to T_c. By Definition (3.3.3) and the remark following it, the algebraic multiplicities of γ relative to T and T_c are equal. Therefore the diagonal multiplicity of γ in $\mathcal{J}$ equals the algebraic multiplicity of γ relative to T for all γ in C with $p(\gamma) \neq 0$. This completes the proof of the theorem.

CHAPTER IV

OPERATORS WITH SPECTRA ON THE UNIT CIRCLE

A theorem of Godement [14] states that, if T is an isometric operator on a complex Banach space X (where $\dim X > 1$), then T has a non-trivial invariant subspace. It follows immediately from this result that, if T is an invertible bounded linear operator on X such that $\|T^n\| \leq M$ $(-\infty < n < \infty)$ for some constant M, then T has a non-trivial invariant subspace. For, if we set

$$\|x\|_1 = \sup\{\|T^n x\| : -\infty < n < \infty\} \qquad (x \in X),$$

it is easy to see that $\| \ \|_1$ is a norm on X which is equivalent to $\| \ \|$, and that T is isometric with respect to $\| \ \|_1$.

In [32], Wermer generalizes this result and obtains an invariant subspace theorem for operators T in $B(X)$ which are invertible and satisfy the following conditions:

(i) $Sp(T)$ contains more than one point;

(ii) there is a sequence $\{a_n\}_{-\infty}^{\infty}$ of real numbers such that

a) $\|T^n\| \leq a_n \qquad (-\infty < n < \infty)$,

b) $a_n = a_{-n} \qquad (-\infty < n < \infty)$,

c) $a_n \uparrow$ and $\dfrac{\log a_n}{n} \downarrow$ as $|n| \to \infty$,

d) $\displaystyle\sum_{-\infty}^{\infty} \frac{\log a_n}{1+n^2} < \infty$.

Condition (ii) implies that

$$\text{(iii)} \quad Sp(T) \subseteq S = \{z \in C : |z| = 1\}.$$

Wermer considers an operator T satisfying (iii) and constructs a family of linear subsets of X , each of which is invariant for T . He then shows that (ii) implies that these subsets are closed, and that (i) implies that at least one of them is non-trivial. In a subsequent paper [34] he shows by example that the order condition on the growth of $\|T^n\|$ as $|n| \rightarrow \infty$ is approximately best possible for his method to work. We remark that Schwartz uses a similar technique in [29] to obtain an invariant subspace theorem for certain operators on Hilbert space with compact imaginary parts.

Our intention is to study operators with spectra lying on the unit circle, and to obtain an invariant subspace theorem which is similar to Wermer's. We consider a certain commutative Banach algebra of sequences and find that condition (ii) can be replaced by demanding that this algebra be completely regular. This algebra was introduced by Wermer in [32], but it was not fully exploited there to construct invariant subspaces, (although our method was mentioned briefly in [34]). Further, we are easily able to extend our results from the complex to the real case.

Finally, we shall assume a knowledge of elementary Banach algebra theory, such as can be found in [24, Chapters I - III]. In particular, we use the Gelfand representation theory of commutative Banach algebras.

1. The algebra $W(T)$.

Throughout this section let X be a complex Banach space and T an invertible operator in $B(X)$. Let

$$S = \{z \in C : |z| = 1\} ,$$

and suppose that

$$Sp(T) \subseteq S .$$

This is equivalent to

$$r(T) = r(T^{-1}) = 1 ,$$

where $r(T)$, $r(T^{-1})$ are the spectral radii of T , T^{-1} respectively.

(4.1.1) Definition. Let $W(T)$ be the set of sequences defined by

$$W(T) = \{h = \{h_n\}_{-\infty}^{\infty} : h_n \in C \text{ and } \sum_{-\infty}^{\infty} |h_n| \|T^n\| < \infty\} .$$

If we define addition and scalar multiplication (by complex numbers) in $W(T)$ coordinate-wise, and set

$$\|h\| = \sum_{-\infty}^{\infty} |h_n| \|T^n\| \qquad (h = \{h_n\} \in W(T)) ,$$

then $W(T)$ is a Banach space.

(4.1.2) Lemma. Let $h = \{h_n\}$ and $k = \{k_n\}$ belong to $W(T)$. Then $\sum_{m=-\infty}^{\infty} h_{n-m} k_m$ is absolutely convergent for all n . Further, if we define the product hk of h and k to be the sequence $\{p_n\}$ given $p_n = \sum_{m=-\infty}^{\infty} h_{n-m} k_m$, then $W(T)$ becomes a complex commutative Banach algebra with identity.

Proof. Since $r(T) = r(T^{-1}) = 1$, $\|T^n\| \geq 1$ for all n . Hence

$$\sum_{-\infty}^{\infty} |h_n| < \infty$$

for each h in $W(T)$. It follows that $\sum_{m=-\infty}^{\infty} h_{n-m} k_m$ is absolutely convergent for all n . Further,

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} |p_n| \|T^n\| &\leq \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} |h_{n-m}| |k_m| \|T^n\| \\
&\leq \sum_{n} \sum_{m} |h_{n-m}| |k_m| \|T^{n-m}\| \|T^m\| \\
&= \{\sum_{n} |h_n| \|T^n\|\}\{\sum_{m} |k_m| \|T^m\|\} \\
&= \|h\| \|k\| \\
&< \infty .
\end{aligned}$$

Thus $p = \{p_n\} \in W(T)$ and $\|p\| \leq \|h\| \|k\|$. It is now elementary to check that $W(T)$ is a complex commutative Banach algebra with identity

e given by

$$e_0 = 1 \ ; \quad e_n = 0 \quad (n \neq 0) \ .$$

Remark. Sequence algebras of this form are considered in [12, Chapter III], where the numbers $\|T^n\|$ are replaced by positive numbers α_n satisfying $\alpha_{n+m} \leq \alpha_n \alpha_m$.

If $h \in W(T)$ then $\sum_n |h_n| < \infty$. Therefore, given λ in S, we can define a function $\phi_\lambda : W(T) \longrightarrow C$ by

$$\phi_\lambda(h) = \sum_{-\infty}^{\infty} h_n \lambda^n \qquad (h = \{h_n\} \in W(T)) \ .$$

(4.1.3) Lemma. The mapping $\lambda \longrightarrow \phi_\lambda$ is a homeomorphism of S on to the carrier space of $W(T)$ (endowed with the usual Gelfand topology).

Proof. A simple calculation shows that, for each $\lambda \in S$, ϕ_λ is a non-zero multiplicative linear functional on $W(T)$. Conversely, suppose that ϕ is a non-zero multiplicative linear functional on $W(T)$. Define $u \in W(T)$ by

$$u_1 = 1 \ ; \quad u_n = 0 \quad (n \neq 1) \ .$$

Then u is regular in $W(T)$ and $\|u^n\| = \|T^n\|$ for all n. Thus

$$0 \neq |\phi(u)|^n = |\phi(u^n)| \leq \|T^n\|$$

for all n. Hence

$$|\phi(u)| \leq \inf_{n>0} \|T^n\|^{\frac{1}{n}} = r(T) = 1 \ ,$$

and

$$|\phi(u)|^{-1} \leq \inf_{n>0} \|T^{-n}\|^{\frac{1}{n}} = r(T^{-1}) = 1 \ .$$

Therefore $|\phi(u)| = 1$, and $\phi(u) = \lambda \in S$. If $h = \{h_n\} \in W(T)$, then $h = \sum_{-\infty}^{\infty} h_n u^n$ in the norm of $W(T)$. Hence

$$\phi(h) = \sum_{-\infty}^{\infty} h_n \lambda^n$$

by the continuity of ϕ; i.e. $\phi = \phi_\lambda$.

Since $\phi_\lambda(u) = \lambda$, $\phi_\lambda = \phi_\mu$ implies that $\lambda = \mu$. Therefore the mapping $\lambda \longrightarrow \phi_\lambda$ is (1,1) and on to the carrier space of $W(T)$. The inverse mapping $\phi \longrightarrow \phi(u)$ is clearly continuous, and hence is a homeomorphism since both spaces are compact and Hausdorff.

Henceforth we shall identify the carrier space of $W(T)$ with S by the above homeomorphism. We write $\hat{h} : S \longrightarrow C$ for the Gelfand transform of h ; i.e.

$$\hat{h}(\lambda) = \sum_{-\infty}^{\infty} h_n \lambda^n \qquad (\lambda \in S) .$$

It is clear that h_n is the nth Fourier coefficient of $\hat{h}$. Therefore $\hat{h} = 0$ implies that $h = 0$. Thus, we have

(4.1.4) Lemma. $W(T)$ is semi-simple.

Given h in $W(T)$,

$$U_h = \sum_{-\infty}^{\infty} h_n T^n$$

defines a bounded linear operator on X , since the infinite sum is absolutely convergent.

(4.1.5) Lemma. The mapping $h \longrightarrow U_h$ is a continuous representation of $W(T)$ on X . Also,

$$Sp(U_h) = \hat{h}(Sp(T))$$

for h in $W(T)$. (Note that $\hat{h}(Sp(T))$ makes sense since $Sp(T) \subseteq S$.)

Proof. That $h \longrightarrow U_h$ is a representation is easy to verify. For instance,

$$\begin{aligned} U_{hk} &= \sum_n \{ \sum_m h_{n-m} k_m \} T^n = \sum_n \sum_m h_{n-m} k_m T^{n-m} T^m \\ &= \{ \sum_n h_n T^n \} \{ \sum_m k_m T^m \} = U_h U_k . \end{aligned}$$

Also,

$$\| U_h \| = \| \sum_n h_n T^n \| \leq \sum_n |h_n| \| T^n \| = \| h \| ,$$

and so the representation is continuous.

Let A be a maximal commutative subalgebra of $B(X)$ containing $\{U_h : h \in W(T)\}$, and let Φ be the carrier space of A. Then

$$\begin{aligned} Sp(U_h) &= Sp_A(U_h) = \{\phi(U_h) : \phi \in \Phi\} \\ &= \{\phi(\sum_n h_n T^n) : \phi \in \Phi\} \\ &= \{\sum_n h_n \{\phi(T)\}^n : \phi \in \Phi\} . \end{aligned}$$

Also,

$$Sp(T) = Sp_A(T) = \{\phi(T) : \phi \in \Phi\} \subseteq S .$$

Therefore

$$\hat{h}\,(Sp(T)) = \{\sum_n h_n \{\phi(T)\}^n : \phi \in \Phi\} = Sp(U_h) .$$

(4.1.6) Definition. Let A be a complex commutative Banach algebra with identity, and let Φ be the carrier space of A. Then A is completely regular if, given a closed subset F of Φ and $\phi \in \Phi \setminus F$, there exists a in A such that

$$\hat{a}\,(\phi) \neq 0 , \quad \hat{a}|F = 0 ,$$

where $\hat{a}$ denotes the Gelfand transform of a.

(4.1.7) Definition. Given $U \in B(X)$, a subspace Y of X is ultra-invariant for U if it is invariant for every V in $B(X)$ with $UV = VU$.

(4.1.8) Theorem. Suppose that $W(T)$ is completely regular and that $Sp(T)$ contains more than one point. Then T has a non-trivial ultra-invariant subspace.

Proof. Let λ_1 and λ_2 be distinct points in $Sp(T)$. Choose an open arc S_1 and a closed arc S_2 in S such that $\lambda_1 \in S_1 \subset S_2$ and $\lambda_2 \notin S_2$. By the complete regularity of $W(T)$, there exist h and k

in $W(T)$ such that

$$\hat{h}(\lambda_1) \neq 0 , \quad \hat{h}|(S \setminus S_1) = 0 ,$$

and

$$\hat{k}(\lambda_2) \neq 0 , \quad \hat{k}|S_2 = 0 .$$

Then $(hk)^\wedge = \hat{h}\hat{k} = 0$; and so, by Lemma (4.1.4), $hk = 0$. Hence

$$U_{hk} = U_h U_k = 0 .$$

By Lemma (4.1.5),

$$0 \neq \hat{h}(\lambda_1) \in \hat{h}(Sp(T)) = Sp(U_h) .$$

Hence $U_h \neq 0$. Similarly, $U_k \neq 0$.

Let $Y = cl(U_k X)$. $Y \neq \{0\}$, since $U_k \neq 0$. If $Y = X$, then

$$U_h(X) = U_h\{cl(U_k X)\} \subseteq cl(U_h U_k X) = \{0\} ,$$

contradicting $U_h \neq 0$. Hence Y is a non-trivial subspace of X . Further, if $V \in B(X)$ and $VT = TV$, then $VU_k = U_k V$. Therefore

$$V(Y) \subseteq cl\{VU_k X\} = cl\ \{U_k V(X)\} \subseteq cl(U_k X) = Y .$$

Thus Y is a non-trivial ultra-invariant subspace for T .

The general question of determining when $W(T)$ is completely regular will be discussed in the next two sections. However, there is one case which is easily disposed of in the following lemma.

(4.1.9) Lemma. If $\|T^n\| = O(|n|^k)$ for some integer $k \geq 0$, then $W(T)$ is completely regular.

Proof. Let S_1 be a closed subset of S , and let $\lambda \in S \setminus S_1$. There exists a function $f : S \longrightarrow C$ such that (a) $f(\lambda) \neq 0$, (b) $f|S_1 = 0$, and (c) if $g : R \longrightarrow C$ is defined by

$$g(x) = f(e^{ix}) \qquad (x \in R) ,$$

then g is $(k + 2)$-times continuously differentiable. We show that

$f = \hat{h}$ for some h in $W(T)$, and this will complete the proof.

For $n \neq 0$, the nth Fourier coefficient h_n of f is given by

$$h_n = \frac{1}{2\pi}\int_{-\pi}^{+\pi} g(x)e^{-inx}dx$$
$$= \frac{1}{2\pi}\int_{-\pi}^{+\pi} g^{(k+2)}(x)\,\frac{e^{-inx}}{(in)^{k+2}}\,dx\,,$$

integrating by parts $k + 2$ times. Thus

$$|h_n| \leq |n|^{-(k+2)}\|g^{(k+2)}\|_\infty = O(|n|^{-(k+2)})\,,$$

and so

$$|h_n|\|T^n\| = O(|n|^{-2})\,.$$

Therefore

$$\sum_{-\infty}^{\infty} |h_n|\|T^n\| < \infty\,,$$

and $h = \{h_n\}$ belongs to $W(T)$. By the remark following Lemma (4.1.3), h_n is the nth Fourier coefficient of $\hat{h}$. Therefore $\hat{h} = f$. This completes the proof.

We now obtain the following result, which is Theorem 3 in [32].

(4.1.10) <u>Theorem</u>. Let T be an invertible operator in $B(X)$, where X is a complex Banach space. Suppose that $\|T^n\| = O(|n|^k)$ for some integer $k \geq 0$. Then either $T = \lambda I$ or T has a non-trivial ultra-invariant subspace.

<u>Proof</u>. Firstly, note that $\|T^n\| = O(|n|^k)$ implies that the spectrum of T lies on the unit circle. Hence we can apply the theory developed above. If the spectrum of T contains more than one point then Theorem (4.1.8), together with Lemma (4.1.9), gives the required result.

Suppose that $Sp(T) = \{\lambda\}$ for some $\lambda \in S$. A theorem of Gelfand and Hille, [30,p.6], gives

$$(T - \lambda I)^{k+1} = 0 .$$

Then, if $T \neq \lambda I$, it is easily seen that

$$\{x : (T - \lambda I)x = 0\}$$

is a non-trivial ultra-invariant subspace for T .

Godement's theorem follows as an immediate corollary.

(4.1.11) Corollary. Let T be an isometric linear operator on a complex Banach space. Then, either $T = \lambda I$ for some λ in S , or T has a non-trivial ultra-invariant subspace.

Proof. Since T is isometric, $T(X)$ is a (closed) subspace of X , and does not equal $\{0\}$. If $T(X) \neq X$, then $T(X)$ is a non-trivial ultra-invariant subspace for T . If $T(X) = X$, T^{-1} is an isometric linear operator and $\|T^n\| = 1$ for all n . Theorem (4.1.10) now gives the required result.

2. Quasi-analytic classes of functions.

Our intention is to investigate conditions on the sequence $\{\|T^n\|\}$ which ensure that $W(T)$ is completely regular. We shall use the theory of quasi-analytic classes of functions and in this section we give a brief account of those parts of that theory which will be needed. For a general discussion of quasi-analyticity, we refer the reader to [6], [21] and [28, Chapter 19].

Let $[a,b]$ be a closed interval in R , and let $M = \{M_k\}_{k \geq 0}$ ba a sequence of positive real numbers.

(4.2.1) Notation. Let $C_M[a,b]$ denote the set of functions f mapping $[a,b]$ into C such that

(i) f is infinitely differentiable on $[a,b]$ (with right and left derivatives of all orders at a and b respectively);

$$\text{(ii)} \quad \|f^{(k)}\|_\infty \leq K_f^k M_k \qquad (k \geq 0) ,$$

where K_f is a constant depending on f but not on k, and $\|f^{(k)}\|_\infty$ denotes the supremum of $|f^{(k)}(x)|$ for x in $[a,b]$.

(4.2.2) Definition. $C_M[a,b]$ is quasi-analytic if

$$f \in C_M[a,b] , \ f^{(k)}(x) = 0 \ \text{ for some } \ x \in [a,b] \ \text{ and all } \ k \geq 0$$

implies that $f = 0$.

The fundamental theorem of the subject is due to Denjoy and Carleman and gives necessary and sufficient conditions on the sequence M for $C_M[a,b]$ to be quasi-analytic. To state it in the form which will be most useful for our purposes, we introduce a function q defined by

$$q(x) = \sup_{k \geq 0} \frac{x^k}{M_k} \qquad (x > 0) .$$

The following properties of q are easily verified.

(i) $\frac{1}{M_0} \leq q(x) \leq \infty \quad (x > 0)$.

(ii) $q(x) \leq q(y)$ if $0 < x \leq y$.

(iii) $q(x) < \infty$ for all $x > 0$ if and only if $(M_k)^{\frac{1}{k}} \longrightarrow \infty$ as $k \longrightarrow \infty$.

(iv) If $q(x) < \infty$ for all x, then q is continuous and $q(x) \longrightarrow \frac{1}{M_0}$ as $x \longrightarrow 0+$.

From (iv), if $q(x) < \infty$ for all x, then

$$I(q) = \int_0^\infty \frac{\log q(x)}{1 + x^2}\, dx$$

either converges or diverges to $+\infty$. If $q(x) = \infty$ eventually, define

$$I(q) = +\infty .$$

We can now state the Denjoy-Carleman theorem (see [21,p.69] and [28,p.376]).

(4.2.3) Theorem.

(i) If $I(q) = +\infty$, then $C_M[a,b]$ is quasi-analytic.

(ii) If $I(q)$ converges, then there exists a function f in $C_M[a,b]$ such that

(a) $f^{(k)}(a) = f^{(k)}(b) = 0 \qquad (k \geq 0)$,

(b) $\|f^{(k)}\|_\infty \leq M_k \qquad (k \geq 0)$,

(c) f is not identically zero .

(4.2.4) Notation. If $f : [-\pi, \pi] \longrightarrow C$ is a continuous function, write

$$f \sim \sum a_n e^{inx}$$

to mean that a_n is the nth Fourier coefficient of f ; i.e.

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx .$$

Suppose that $-\pi < a < b < \pi$ and that M and q are as above. Then we have the following result, which is essentially due to Mandelbrojt [21,pp.78 et seq.].

(4.2.5) Theorem.

(i) Suppose that $I(q) = +\infty$, and that $f : [-\pi, \pi] \longrightarrow C$ is infinitely differentiable and vanishes outside $[a,b]$. Suppose further that

$$|a_n| q(|n|) = O(1) ,$$

where $f \sim \sum a_n e^{inx}$. Then $f = 0$.

(ii) Suppose that $I(q) < \infty$. Then there exists a function $f : [-\pi, \pi] \longrightarrow C$ such that

(a) f is infinitely differentiable;

(b) f vanishes outside $[a,b]$, but is not identically zero;

(c) $|a_n| q(|n|) = O(1)$, where $f \sim \sum a_n e^{inx}$.

<u>Proof</u>. (i) Suppose that $I(q)$ and f are as in (i) in the statement of the theorem. Since f is infinitely differentiable, it is easy to show that

$$f^{(k)}(x) = \sum_{-\infty}^{\infty} (in)^k a_n e^{inx} \qquad (-\pi \leq x \leq \pi \ , \ k \geq 0) \ ,$$

where the right hand side is uniformly absolutely convergent. Thus

$$\|f^{(k)}\|_{\infty} \leq \sum_{n=-\infty}^{\infty} |n|^k |a_n| \qquad (k \geq 0) \ .$$

Also, from the definition of q,

$$q(|n|) \geq \frac{|n|^k}{M_k} \qquad (k \geq 0) \ .$$

Since $|a_n| q(|n|) = O(1)$, it follows that

$$\|f^{(k)}\|_{\infty} \leq K_1 \sum_{n=-\infty}^{\infty} \frac{|n|^k M_{k+2}}{|n|^{k+2}} \qquad (k \geq 0)$$

for some constant K_1. Thus

$$\|f^{(k)}\|_{\infty} \leq K_2 M_{k+2} \qquad (k \geq 0)$$

for some constant K_2. Define $g : [-\pi, \pi] \longrightarrow C$ by

$$g(x) = \int_{-\pi}^{x} ds \int_{-\pi}^{s} f(t) dt \qquad (-\pi \leq x \leq \pi) \ .$$

Then g is infinitely differentiable and

$$g^{(k+2)} = f^{(k)} \qquad (k \geq 0) \ .$$

Hence $g \in C_M[-\pi, \pi]$. Also, since f vanishes on $[-\pi, a]$,

$$g^{(k)}(x) = 0 \qquad (-\pi \leq x \leq a \ , \ k \geq 0) \ .$$

Therefore, by Theorem (4.2.3) (i) and Definition (4.2.2), $g = 0$.

Hence $f = 0$.

(ii) Suppose that $I(q) < \infty$. By Theorem (4.2.3) (ii), there exists an infinitely differentiable function $g : [a,b] \longrightarrow C$ such that

$$\text{(a)}\quad g^{(k)}(a) = g^{(k)}(b) = 0 \qquad (k \geq 0) ;$$

$$\text{(b)}\quad \|g^{(k)}\|_{\infty} \leq M_k \qquad (k \geq 0) ;$$

(c) g is not identically zero.

Define $f : [-\pi,\pi] \longrightarrow C$ by

$$f(x) = g(x) \qquad (a \leq x \leq b) ,$$

$$f(x) = 0 \qquad (-\pi \leq x < a ,\ b < x \leq \pi) .$$

Then f is not identically zero, vanishes outside $[a,b]$, and is infinitely differentiable. Suppose that $f \sim \sum a_n e^{inx}$. For $n \neq 0$ and $k \geq 0$,

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f^{(k)}(x)\,\frac{e^{-inx}}{(in)^k}\,dx .$$

Therefore

$$|a_n| \leq \frac{\|f^{(k)}\|_{\infty}}{|n|^k} = \frac{\|g^{(k)}\|_{\infty}}{|n|^k} \leq \frac{M_k}{|n|^k}$$

for $n \neq 0$ and $k \geq 0$. Hence

$$|a_n| \leq \inf_{k \geq 0}\frac{M_k}{|n|^k} = \left\{\sup_{k \geq 0}\frac{|n|^k}{M_k}\right\}^{-1} = \frac{1}{q(|n|)} .$$

Therefore

$$|a_n|q(|n|) = O(1) ,$$

and the proof is complete.

We now give a characterization of those continuous functions $q : [0,\infty) \longrightarrow (0,\infty)$ for which there is a sequence $\{M_k\}_{k \geq 0}$ of positive numbers such that

$$q(x) = \sup_{k \geq 0}\frac{x^k}{M_k} \qquad (x > 0) ,$$

in terms of their values at $0,1,2,\ldots$.

(4.2.6) Theorem.

(i) Suppose that $\{M_k\}_{k \geq 0}$ is a sequence of positive numbers such that $q(x) = \sup_{k \geq 0} \frac{x^k}{M_k} < \infty$ for $x > 0$. Define $q(0) = \frac{1}{M_0}$.

Then

(a) $0 < q(n) \leq q(n+1) \qquad (n \geq 0)$;

(b) there is a sequence $\{k(n)\}_{n \geq 1}$ of non-negative integers such that $k(n) \uparrow \infty$ as $n \longrightarrow \infty$, and

$$k(n) \leq \frac{\log q(n+1) - \log q(n)}{\log (n+1) - \log n} \leq k(n+1) \qquad (n \geq 1) .$$

Conversely,

(ii) suppose that $\{a_n\}_{n \geq 0}$ is a sequence of real numbers such that

(a) $0 < a_n \leq a_{n+1} \qquad (n \geq 0)$;

(b) there is a sequence $\{k(n)\}_{n \geq 1}$ of non-negative integers such that $k(n) \uparrow \infty$ as $n \longrightarrow \infty$, and

$$k(n) \leq \frac{\log a_{n+1} - \log a_n}{\log (n+1) - \log n} \leq k(n+1) \qquad (n \geq 1) .$$

Then there is a sequence $\{M_k\}_{k \geq 0}$ of positive numbers such that

$$a_0 = \frac{1}{M_0} , \; a_n = \sup_{k \geq 0} \frac{n^k}{M_k} \qquad (n \geq 1) .$$

Proof. (i) Let $\{M_k\}_{k \geq 0}$ and q be as in the statement of part (i) of the theorem. From remark (iii) on p.59, $(M_k)^{\frac{1}{k}} \longrightarrow \infty$ as $k \longrightarrow \infty$. Hence $\frac{x^k}{M_k} \longrightarrow 0$ as $k \longrightarrow \infty$, for all $x > 0$.

Therefore

$$q(x) = \max_{k \geq 0} \frac{x^k}{M_k}$$

for $x > 0$. For $n \geq 1$, let $k(n)$ be the greatest integer k such that

$$q(n) = \frac{n^k}{M_k} .$$

Then

$$\frac{n^{k(n)}}{M_{k(n)}} \geq \frac{n^{k(n+1)}}{M_{k(n+1)}} \quad \text{and} \quad \frac{(n+1)^{k(n+1)}}{M_{k(n+1)}} \geq \frac{(n+1)^{k(n)}}{M_{k(n)}}$$

for $n \geq 1$. Hence

$$\{k(n+1) - k(n)\} \log n \leq \log M_{k(n+1)} - \log M_{k(n)} \tag{1}$$

and

$$\log M_{k(n+1)} - \log M_{k(n)} \leq \{k(n+1) - k(n)\} \log (n+1) \tag{2}$$

for $n \geq 1$. Also,

$$q(n) = \frac{n^{k(n)}}{M_{k(n)}} \qquad (n \geq 1) ,$$

and so

$$\log q(n+1) - \log q(n) = k(n+1)\log (n+1) - k(n)\log n + \log M_{k(n)} - \log M_{k(n+1)} \tag{3}$$

for $n \geq 1$. (1), (2) and (3) give

$$k(n) \leq \frac{\log q(n+1) - \log q(n)}{\log (n+1) - \log n} \leq k(n+1) \qquad (n \geq 1) .$$

It is now easy to verify that q and $\{k(n)\}$ satisfy properties (a) and (b).

(ii) Let $\{a_n\}_{n \geq 0}$ and $\{k(n)\}_{n \geq 1}$ satisfy the conditions in the statement of part (ii) of the theorem. Let n_1 be the smallest integer n (> 0) such that $k(n) > 0$. Define $M_0 = \frac{1}{a_0}$ and

$$M_k = \frac{n_1^{k(n_1)}}{a_{n_1}}$$

for $1 \leq k \leq k(n_1)$. If $k > k(n_1)$, there is a unique n ($> n_1$) such that $k(n-1) < k \leq k(n)$. Put

$$M_k = \frac{n^{k(n)}}{a_n} .$$

Thus, if we define $k(0) = 0$, then

$$M_o = \frac{1}{a_o} \text{ and } M_k = \frac{n^{k(n)}}{a_n} ,$$

where $k(n - 1) < k \leq k(n)$.

Let $n \geq 1$ be fixed. If $k(m - 1) < k \leq k(m)$ then $M_k = M_{k(m)}$. Therefore $\frac{n^k}{M_k} \leq \frac{n^{k(m)}}{M_{k(m)}}$. It follows that

$$\sup_{k \geq 0} \frac{n^k}{M_k} = \sup_{m \geq 0} \frac{n^{k(m)}}{M_{k(m)}} . \tag{4}$$

Now

$$\frac{(n + 1)^{k(n+1)}}{n^{k(n+1)}} \geq \frac{a_{n+1}}{a_n} ,$$

and so

$$\frac{n^{k(n+1)}}{M_{k(n+1)}} = \frac{n^{k(n+1)}}{(n + 1)^{k(n+1)}} a_{n+1} \leq a_n = \frac{n^{k(n)}}{M_{k(n)}} . \tag{5}$$

We show, by induction on m , that

$$\frac{n^{k(n+m)}}{M_{k(n+m)}} \leq a_n \qquad (m \geq 1) . \tag{6}$$

The case $m = 1$ is given by (5). Also, replacing n by $n+m$ in (5), we get

$$\frac{(n + m)^{k(n+m+1)}}{M_{k(n+m+1)}} \leq \frac{(n + m)^{k(n+m)}}{M_{k(n+m)}} . \tag{7}$$

Suppose that (6) holds for some $m \geq 1$. Then

$$\begin{aligned} \frac{n^{k(n+m+1)}}{M_{k(n+m+1)}} &= \frac{n^{k(n+m)}}{M_{k(n+m)}} \frac{M_{k(n+m)}}{M_{k(n+m+1)}} \frac{n^{k(n+m+1)}}{n^{k(n+m)}} \\ &\leq \frac{n^{k(n+m)}}{M_{k(n+m)}} \frac{M_{k(n+m)}}{M_{k(n+m+1)}} \frac{(n + m)^{k(n+m+1)}}{(n + m)^{k(n+m)}} , \end{aligned}$$

since $k(n+m+1) \geq k(n+m)$. Hence, by (7),

$$\begin{aligned} \frac{n^{k(n+m+1)}}{M_{k(n+m+1)}} &\leq \frac{n^{k(n+m)}}{M_{k(n+m)}} \\ &\leq a_n , \text{ by the induction hypothesis .} \end{aligned}$$

Therefore, by induction, (6) holds.

Since $a_n = \dfrac{n^{k(n)}}{M_{k(n)}}$, equations (4) and (6) give

$$\sup_{k \geq 0} \frac{n^k}{M_k} = \sup_{0 \leq m \leq n} \frac{n^{k(m)}}{M_{k(m)}} \quad . \tag{8}$$

Using the fact that

$$\frac{a_{m+1}}{a_m} \geq \frac{(m+1)^{k(m)}}{m^{k(m)}} \qquad (m \geq 1) \ ,$$

it can be shown, in the same way that (6) was proved, that

$$\frac{n^{k(m)}}{M_{k(m)}} \leq a_n = \frac{n^{k(n)}}{M_{k(n)}} \tag{9}$$

if $1 \leq m \leq n$. Also,

$$\frac{n^{k(0)}}{M_{k(0)}} = a_o \leq a_n \ . \tag{10}$$

Hence, from (8),(9) and (10) ,

$$a_n = \sup_{k \geq 0} \frac{n^k}{M_k} \ ,$$

This completes the proof of (ii).

(4.2.7) <u>Definition</u>. A sequence $\{a_n\}_{n \geq 0}$ of real numbers is <u>log-convex</u> if

$$\text{(i)} \quad 0 < a_n \leq a_{n+1} \qquad (n \geq 0) \ ,$$

$$\text{(ii)} \quad \frac{\log a_{n+1} - \log a_n}{\log (n+1) - \log n} \uparrow \infty \text{ as } n \longrightarrow \infty \ .$$

We can now obtain the theorem which will be used in the next section to give conditions which will ensure that $W(T)$ is completely regular. The result is essentially Theorems I,II in [21,Chapter VI].

(4.2.8) <u>Theorem</u>. Let $a,b \in R$ be such that $-\pi < a < b < \pi$, and let $\{a_n\}_{n \geq 0}$ be a log-convex sequence of real numbers.

(i) Suppose that $\sum_{n=1}^{\infty} \frac{\log a_n}{n^2} = \infty$, and that $f : [-\pi, \pi] \longrightarrow C$ is an infinitely differentiable function which vanishes ouside $[a,b]$. Suppose further that

$$|c_n| a_{|n|} = O(1) ,$$

where $f \sim \sum c_n e^{inx}$. Then $f = 0$.

(ii) Suppose that $\sum_{n=1}^{\infty} \frac{\log a_n}{n^2} < \infty$. Then there exists a function $f : [-\pi, \pi] \longrightarrow C$ such that

(a) f is infinitely differentiable;

(b) f vanishes outside $[a,b]$, but is not identically zero;

(c) $|c_n| a_{|n|} = O(1)$, where $f \sim \sum c_n e^{inx}$.

<u>Proof</u>. For $n \geq 1$, let $N(n) =$ the integral part of $\dfrac{\log a_{n+1} - \log a_n}{\log (n+1) - \log n}$.

Then

$$0 \leq N(n) \text{ for all } n \text{, and } N(n) \uparrow \infty \text{ as } n \longrightarrow \infty . \tag{11}$$

Define sequences $\{b_n\}_{n \geq 0}$, $\{b'_n\}_{n \geq 0}$ as follows :

$$b_0 = a_0 \, , \; b_1 = a_1 \, , \; b_{n+1} = b_n \exp \left\{ N(n) \log \frac{n+1}{n} \right\} \qquad (n \geq 1) ;$$

$$b'_0 = a_0 \, , \; b'_n = n b_n \qquad (n \geq 1) .$$

The following properties of these sequences are easily verified.

$$\frac{\log b_{n+1} - \log b_n}{\log (n+1) - \log n} = N(n) \geq 0 \qquad (n \geq 1) . \tag{12}$$

$$\frac{\log b'_{n+1} - \log b'_n}{\log (n+1) - \log n} = N(n) + 1 > 0 \qquad (n \geq 1) . \tag{13}$$

$$b_n \leq a_n \leq b'_n \; ; \; \log b'_n = \log b_n + \log n \qquad (n \geq 1) . \tag{14}$$

From (11),(12) and (13), we see that the sequences $\{b_n\}_{n \geq 0}$ and $\{b'_n\}_{n \geq 0}$ satisfy conditions (a) and (b) in Theorem (4.2.6) (ii).

Hence we can find sequences $\{M_n\}$ and $\{M_n'\}$, and corresponding functions q and q' such that

$$b_n = q(n) , \; b_n' = q'(n) \tag{15}$$

for $n \geq 1$.

(i) Suppose that $\sum_{n=1}^{\infty} \frac{\log a_n}{n^2} = \infty$ and that f is as in the statement of part (i) of the theorem. From (14) and (15), we see that

$$|c_n| b_{|n|} = |c_n| q(|n|) = O(1) ,$$

and that

$$I(q) = \infty .$$

Hence, by Theorem (4.2.5) (i), $f = 0$.

(ii) Suppose that $\sum_{n=1}^{\infty} \frac{\log a_n}{n^2} < \infty$. It follows from (14) that $\sum_{n=1}^{\infty} \frac{\log b_n'}{n^2} < \infty$, and hence, by (15), that

$$I(q') < \infty .$$

Therefore, by Theorem (4.2.5) (ii), there exists a function $f : [-\pi, \pi] \longrightarrow C$ such that

(a) f is infinitely differentiable;

(b) f vanishes outside $[a,b]$, but is not identically zero;

(c) $|c_n| q'(|n|) = O(1)$, where $f \sim \sum c_n e^{inx}$.

It follows from (14) and (15) that

$$|c_n| a_{|n|} = O(1) ,$$

and this completes the proof of (ii).

3. Complete regularity.

We now return to the situation discussed in Section 1. Throughout this section, X is a complex Banach space and T is a bounded linear operator on X such that

$$Sp(T) \subseteq S = \{z \in C : |z| = 1\} .$$

Further, $W(T)$ is the sequence algebra defined by (4.1.1) and studied in Section 1. We shall investigate conditions under which $W(T)$ is completely regular. From the identification of the carrier space of $W(T)$ with S given by Lemma (4.1.3), and from the definition of complete regularity given by (4.1.6), the following result is clear.

(4.3.1) Lemma. $W(T)$ is completely regular if and only if, given λ in S and a closed subset S_1 of S with $\lambda \notin S_1$, there is a continuous function $f : S \longrightarrow C$ such that

$$\text{(i)} \quad f(\lambda) \neq 0 , \ f|S_1 = 0 ;$$
$$\text{(ii)} \quad \sum_{-\infty}^{\infty} |c_n| \|T^n\| < \infty , \text{ where } f \sim \sum c_n e^{inx} .$$

Remark. By the Riemann-Lebesgue lemma, the Fourier coefficients of any continuous function tend to 0 as $|n| \longrightarrow \infty$. Roughly speaking, to ask for complete regularity is to ask for the existence, for each proper closed subset S_1 of S, of a continuous function f which vanishes on S_1 and whose Fourier coefficients tend to 0 fast enough for (ii) to hold, and yet is not identically zero. Hence $W(T)$ will be completely regular if the sequence $\{\|T^n\|\}$ does not diverge too rapidly as $|n| \longrightarrow \infty$, (e.g. , by Lemma (4.1.9), if $\|T^n\| = O(|n|^k)$). In this section we attempt to give a more precise meaning to 'too rapidly' in this context.

Lemma (4.3.1) can in fact be improved upon in the following way.

(4.3.2) Lemma. For $n \geq 0$, let $\alpha_n = \max\{\|T^n\|, \|T^{-n}\|\}$. Then $W(T)$ is completely regular if and only if, given ε with $0 < \varepsilon < \pi$, there exists an infinitely differentiable function $f : [-\pi,\pi] \longrightarrow C$ such that

(i) f is not identically zero;

(ii) f vanishes outside $[-\varepsilon,\varepsilon]$;

(iii) $|c_n|\alpha_{|n|} = O(1)$, where $f \sim \sum c_n e^{inx}$.

Proof. Firstly, we make several remarks on the convolution product of two functions. Let f and g be continuous functions mapping $[-\pi,\pi]$ into C , with $f(-\pi) = f(\pi)$ and $g(-\pi) = g(\pi)$. If we also denote by f , g the periodic extensions (of period 2π) of f , g respectively, we can define the convolution product $f*g : [-\pi,\pi] \longrightarrow C$ by

$$(f*g)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)g(t)dt \qquad (-\pi \leq x \leq \pi) .$$

$f*g$ is continuous and $(f*g)(-\pi) = (f*g)(\pi)$. The following results are standard. (See, e.g., [19,Chapter 2] and [27,Chapter 1].)

(a) If $f \sim \sum a_n e^{inx}$ and $g \sim \sum b_n e^{inx}$, then $f*g \sim \sum a_n b_n e^{inx}$.

(b) If f vanishes outside $[-\varepsilon,\varepsilon]$ and g vanishes outside $[-\eta,\eta]$, where $\varepsilon + \eta < \pi$, then $f*g$ vanishes outside $[-(\varepsilon+\eta), \varepsilon+\eta]$.

Also, by a consideration of Fourier series, it is easy to show that

(c) if f is infinitely differentiable and $f^{(k)}(-\pi) = f^{(k)}(\pi)$ for all $k \geq 0$, then $f*g$ is infinitely differentiable and $(f*g)^{(k)} = f^{(k)}*g$ for all $k \geq 0$.

We now proceed to prove the lemma. Suppose that $W(T)$ is completely regular and that $0 < \varepsilon < \pi$. Let S_1 be the closed subset of S given by

$$S_1 = \{e^{ix} : x \in [-\pi,\pi] \setminus (-\tfrac{\varepsilon}{4}, \tfrac{\varepsilon}{4})\} .$$

There exists $h = \{h_n\}$ in $W(T)$ such that $\hat{h} \mid S_1 = 0$ and $\hat{h}(1) \neq 0$. Since $\hat{h} \neq 0$, $h_{n_o} \neq 0$ for some n_o. Define $f_1 : [-\pi,\pi] \longrightarrow C$ by

$$f_1(x) = e^{-in_o x}\, \hat{h}(e^{ix}) \qquad (-\pi \leq x \leq \pi) .$$

Then f_1 vanishes outside $[-\frac{\varepsilon}{4}, \frac{\varepsilon}{4}]$ and $f_1 \sim \sum h_{n+n_o} e^{inx}$. Define $f_2 : [-\pi,\pi] \longrightarrow C$ by

$$f_2(x) = f_1(-x) \qquad (-\pi \leq x \leq \pi) .$$

Then f_2 vanishes outside $[-\frac{\varepsilon}{4}, \frac{\varepsilon}{4}]$ and $f_2 \sim \sum h_{-n+n_o} e^{inx}$. Let $f_3 = f_1 * f_2 : [-\pi,\pi] \longrightarrow C$. By remarks (a) and (b) above, f_3 vanishes outside $[-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]$ and $f_3 \sim \sum h_{n+n_o} h_{-n+n_o} e^{inx}$. Let $f_4 : [-\pi,\pi] \longrightarrow C$ be an infinitely differentiable function which vanishes outside $[-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]$ but is not identically zero. Suppose that $f_4 \sim \sum a_n e^{inx}$. By multiplying f_4 by an appropriate power of e^{ix} if necessary, we can assume that $a_o \neq 0$. Let $f = f_3 * f_4$. By remark (c), f is an infinitely differentiable function mapping $[-\pi,\pi]$ into C; and, by (b), f vanishes outside $[-\varepsilon,\varepsilon]$. From (a), $f \sim \sum c_n e^{inx}$ where $c_n = a_n h_{n+n_o} h_{-n+n_o}$. Therefore $c_o = a_o h_{n_o}^2 \neq 0$, and so f is not identically zero. Also

$$\begin{aligned} |h_{n+n_o}||h_{-n+n_o}|\alpha_{|n|} &\leq |h_{n+n_o}| \|T^n\| \, |h_{-n+n_o}| \|T^{-n}\| \\ &\leq |h_{n+n_o}| \|T^{n+n_o}\| \, |h_{-n+n_o}| \|T^{-n+n_o}\| \|T^{-n_o}\|^2 \\ &= O(1) , \end{aligned}$$

since $\sum_{-\infty}^{\infty} |h_n| \|T^n\| < \infty$. Further, $|a_n| = O(1)$. Hence

$$|c_n|\alpha_{|n|} = |a_n h_{n+n_o} h_{-n+n_o}|\alpha_{|n|} = O(1) .$$

Therefore f satisfies conditions (i) - (iii), and the first part of the lemma is proved.

Suppose conversely that, for each ε with $0 < \varepsilon < \pi$, there is an infinitely differentiable function $f : [-\pi,\pi] \longrightarrow C$ satisfying conditions (i) - (iii). We show that $W(T)$ is completely regular.

Let S_1 be a closed subset of S, and let $\lambda \in S \setminus S_1$. Let $\lambda = e^{ix_0}$ where $-\pi \leq x_0 < \pi$, and choose ε such that $0 < \varepsilon < \pi$ and

$$\{e^{ix} : x_0 - \varepsilon \leq x \leq x_0 + \varepsilon\} \cap S_1 = \emptyset .$$

Let $f_1 : [-\pi, \pi] \longrightarrow C$ be a function which vanishes outside $[-\frac{\varepsilon}{4}, \frac{\varepsilon}{4}]$, is not identically zero, and is differentiable with continuous second derivative. As in the proof of Lemma (4.1.9), it is elementary to show that, if $f_1 \sim \sum a_n e^{inx}$, then $|a_n| = O(|n|^{-2})$. Since f_1 is not identically zero, $a_{n_0} \neq 0$ for some n_0.

By hypothesis, there is an infinitely differentiable function $f_2 : [-\pi,\pi] \longrightarrow C$ such that (i) f_2 is not identically zero; (ii) f_2 vanishes outside $[-\frac{\varepsilon}{4}, \frac{\varepsilon}{4}]$; and (iii) $|b_n|\alpha_{|n|} = O(1)$, where $f_2 \sim \sum b_n e^{inx}$. Since f_2 is not identically zero, $b_{n_1} \neq 0$ for some n_1. Let

$$f_3(x) = e^{i(n_0-n_1)x} f_2(x) \qquad (-\pi \leq x \leq \pi) .$$

Then f_3 vanishes outside $[-\frac{\varepsilon}{4}, \frac{\varepsilon}{4}]$ and $f_3 \sim \sum c_n e^{inx}$, where $c_n = b_{n+n_1-n_0}$. Let $f_4 = f_1 * f_3$. Then f_4 vanishes outside $[-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]$ and $f_4 \sim \sum d_n e^{inx}$, where

$$d_n = a_n c_n = a_n b_{n+n_1-n_0} .$$

Hence $d_{n_0} = a_{n_0} b_{n_1} \neq 0$, and so f_4 is not identically zero. Also,

$$|c_n| \|T^n\| = |b_{n+n_1-n_o}| \|T^n\| \leq |b_{n+n_1-n_o}| \|T^{n+n_1-n_o}\| \|T^{n_o-n_1}\|$$

$$\leq \alpha_{|n+n_1-n_o|} |b_{n+n_1-n_o}| \|T^{n_o-n_1}\| = O(1) \ ;$$

and

$$|a_n| = O(|n|^{-2}) \ .$$

Therefore

$$\sum_{-\infty}^{\infty} |d_n| \|T^n\| = \sum_{-\infty}^{\infty} |a_n| |c_n| \|T^n\| < \infty .$$

Since f_4 is not identically zero, there exists x_1 with $-\frac{\varepsilon}{2} < x_1 < \frac{\varepsilon}{2}$ such that $f_4(x_1) \neq 0$. Define $f : S \longrightarrow C$ by

$$f(e^{ix}) = f_4(x + x_1 - x_o) \qquad (-\pi \leq x < \pi) \ .$$

Then f is continuous and vanishes outside

$$\{e^{ix} : x_o - x_1 - \frac{\varepsilon}{2} \leq x \leq x_o - x_1 + \frac{\varepsilon}{2}\} \ .$$

But $|x_1| < \frac{\varepsilon}{2}$, and so f vanishes outside

$$\{e^{ix} : x_o - \varepsilon < x < x_o + \varepsilon\} \ .$$

Therefore $f|S_1 = 0$. Also,

$$f(\lambda) = f(e^{ix_o}) = f_4(x_1) \neq 0 \ .$$

Finally, $f \sim \sum d_n e^{in(x_1-x_o)} e^{inx}$, and

$$\sum_{-\infty}^{\infty} |d_n e^{in(x_1-x_o)}| \|T^n\| = \sum_{-\infty}^{\infty} |d_n| \|T^n\| < \infty .$$

Hence the sequence $h = \{d_n e^{in(x_1-x_o)}\}$ belongs to $W(T)$. Further, $\hat{h} = f$; and so

$$\hat{h}(\lambda) \neq 0 \ , \ \hat{h}|S_1 = 0 \ .$$

Thus $W(T)$ is completely regular.

We now obtain a condition on the growth of $\|T^n\|$ which will ensure that $W(T)$ is completely regular.

(4.3.3) <u>Theorem</u>. For $n \geq 0$, let $\alpha_n = \max\{\|T^n\|, \|T^{-n}\|\}$. If there exists a log-convex sequence $\{a_n\}_{n \geq 0}$ such that

$$\text{(i)} \quad \alpha_n \leq a_n \qquad (n \geq 0),$$

$$\text{(ii)} \quad \sum_{n=1}^{\infty} \frac{\log a_n}{n^2} < \infty,$$

then $W(T)$ is completely regular.

<u>Proof</u>. Let $0 < \varepsilon < \pi$. By Theorem (4.2.8) (ii), there exists a function $f : [-\pi, \pi] \longrightarrow C$ such that (a) f is infinitely differentiable; (b) f vanishes outside $[-\varepsilon, \varepsilon]$, but is not identically zero; and (c) $|c_n| a_{|n|} = O(1)$, where $f \sim \sum c_n e^{inx}$. Then

$$|c_n| \alpha_{|n|} = O(1),$$

and the result now follows from Lemma (4.3.2).

The following invariant subspace theorem is immediate from this result, together with Theorem (4.1.8).

(4.3.4) <u>Theorem</u>. Let T be an invertible bounded linear operator on a complex Banach space X. Suppose that $Sp(T)$ contains more than one point and that there is a log-convex sequence $\{a_n\}_{n \geq 0}$ such that

$$\text{(i)} \quad \|T^n\| \leq a_{|n|} \qquad (-\infty < n < \infty),$$

$$\text{(ii)} \quad \sum_{n=1}^{\infty} \frac{\log a_n}{n^2} < \infty.$$

Then T has a non-trivial ultra-invariant subspace.

<u>Proof</u>. For $n \geq 0$,

$$a_n^2 \geq \|T^n\| \|T^{-n}\| \geq 1;$$

and so $a_n \geq 1$. Hence $\log a_n \geq 0$ for all n. If $\varlimsup \frac{\log a_n}{n} > 0$, then $\sum_{n=1}^{\infty} \frac{\log a_n}{n^2} = \infty$, contradicting (ii). Therefore

$$\varlimsup \frac{\log a_n}{n} = 0.$$

It follows from this, together with (i), that

$$r(T) = r(T^{-1}) = 1 ,$$

and hence that $Sp(T) \subseteq S$. Also, condition (i) gives

$$\max \{\|T^n\|, \|T^{-n}\|\} \leq a_n \qquad (n \geq 0) .$$

The result now follows from Theorems (4.1.8) and (4.3.3).

Remark. The above result is closely related to the theorem which was proved by Wermer in [32] and which is stated on p.50. Although Wermer does not do so, the non-trivial invariant subspace which he constructs can easily be shown to be ultra-invariant. With this, Wermer's theorem can be restated in the following way.

Theorem. Let T be in invertible bounded linear operator on a complex Banach space X . Suppose that $Sp(T)$ contains more than one point, and that there is a sequence $\{a_n\}_{n \geq 0}$ of positive numbers, with $a_n \uparrow$ and $\frac{\log a_n}{n} \downarrow$ as $n \longrightarrow \infty$, such that

$$\text{(i)} \quad \|T^n\| \leq a_{|n|} \qquad (-\infty < n < \infty) ,$$

$$\text{(ii)} \quad \sum_{n=1}^{\infty} \frac{\log a_n}{n^2} < \infty .$$

Then T has a non-trivial ultra-invariant subspace.

Thus, the only difference between the two theorems is that in one $\|T^n\|$ is dominated by a log-convex sequence, and in the other $\|T^n\|$ is dominated by a sequence $\{a_n\}$ with $a_n \uparrow$ and $\frac{\log a_n}{n} \downarrow$. We have been unable to show that, if $\|T^n\| \leq a_{|n|}$, where $\{a_n\}$ is a log-convex sequence with $\sum_{n=1}^{\infty} \frac{\log a_n}{n^2} < \infty$, then $\|T^n\| \leq b_{|n|}$, where $\{b_n\}$ is a sequence satisfying $b_n \uparrow$, $\frac{\log b_n}{n} \downarrow$ and $\sum_{n=1}^{\infty} \frac{\log b_n}{n^2} < \infty$; or vice versa. It therefore remains undecided whether one of the above theorems is a generalization of the other, or whether, though similar, they are in fact independent.

We end this section with a theorem which is a partial converse to Theorem (4.3.3). It gives an indication of the extent to which Theorem (4.1.8) can be used to show the existence of non-trivial invariant subspaces.

(4.3.5) <u>Theorem</u>. For $n \geq 0$, let $\alpha_n = \max\{\|T^n\|, \|T^{-n}\|\}$. Suppose that there is a log-convex sequence $\{a_n\}_{n \geq 0}$ such that

$$\text{(i)}\ \alpha_n \geq a_n \qquad (n \geq 0)\ ,$$

$$\text{(ii)}\ \sum_{n=1}^{\infty} \frac{\log a_n}{n^2} = \infty\ .$$

Then $W(T)$ is not completely regular.

<u>Proof</u>. Suppose that $W(T)$ is completely regular. By Lemma (4.3.2), there is an infinitely differentiable function $f : [-\pi, \pi] \longrightarrow C$ such that

(i) f is not identically zero;

(ii) f vanishes outside $[-\frac{\pi}{2}, \frac{\pi}{2}]$;

(iii) $|c_n|\alpha_{|n|} = O(1)$, where $f \sim \sum c_n e^{inx}$.

Then $|c_n|a_{|n|} = O(1)$. But, by Theorem (4.2.8) (i), this implies that $f = 0$, which is an obvious contradiction. Hence $W(T)$ is not completely regular.

4. <u>Real spaces</u>.

The results of the previous sections can easily be extended to real Banach spaces and corresponding invariant subspace theorems obtained. Throughout this section, let X be a real Banach space and T

an invertible bounded linear operator on X. Let T_c denote the usual extension of T to the complexification X_c of X. It is clear that T_c is invertible and that $(T_c)^{-1} = (T^{-1})_c$. More generally,

$$(T_c)^n = (T^n)_c \qquad (-\infty < n < \infty) .$$

Hence $\|(T_c)^n\| = \|(T^n)_c\| = \|T^n\|$ for all n, from which it follows that $r(T) = r(T_c)$ and $r(T^{-1}) = r((T_c)^{-1})$. We shall assume that

$$r(T) = r(T^{-1}) = 1 .$$

Following Definition (4.1.1), we define

$$W(T) = \{h = \{h_n\}_{-\infty}^{\infty} : h_n \in C \text{ and } \sum_{-\infty}^{\infty} |h_n| \|T^n\| < \infty\} .$$

We note that, by the above remarks, $W(T) = W(T_c)$. Hence $W(T)$ is again a semi-simple complex commutative Banach algebra, with carrier space homeomorphic to S.

(4.4.1) Theorem. If $Sp(T)$ contains two points which are not complex conjugates of each other, and if $W(T)$ is completely regular, then T has a non-trivial ultra-invariant subspace.

Proof. As remarked on p.4, $Sp(T)$ is a self-conjugate subset of C. Therefore we can suppose that $e^{ix_1}, e^{ix_2} \in Sp(T)$, where $0 \leq x_1 < x_2 \leq \pi$. We show that there exist sequences $h = \{h_n\}$ and $k = \{k_n\}$ in $W(T)$ such that

(i) h_n and k_n are real for all n;

(ii) $hk = 0$;

(iii) $\hat{h}(e^{ix_1}) \neq 0$ and $\hat{k}(e^{ix_2}) \neq 0$.

There are three cases to be considered.

Case (a). Suppose that $x_1 = 0$ and $x_2 = \pi$. Let

$$S_1 = \{e^{ix} : |x| < \tfrac{\pi}{2}\}$$

and

$$S_2 = \{e^{ix} : \tfrac{\pi}{2} < |x| \leq \pi\} .$$

Since $W(T)$ is completely regular, there is a sequence $h^1 = \{h_n^1\}$ in $W(T)$ such that

$$\hat{h}^1 | S \setminus S_1 = 0 \text{ and } \hat{h}^1(e^{ix_1}) = \hat{h}^1(1) \neq 0 .$$

Further, by multiplying h^1 by a suitable constant, we can assume that $\hat{h}^1(1)$ is real. Define the sequence $h^2 = \{h_n^2\}$ by

$$h_n^2 = \overline{h_n^1} \qquad (-\infty < n < \infty) ,$$

where $\bar{z}$ denotes the complex conjugate of z in C. Clearly $h^2 \in W(T)$, and

$$\hat{h}^2(\lambda) = \sum_{-\infty}^{\infty} \overline{h_n^1} \lambda^n = \overline{\hat{h}^1(\bar{\lambda})} \qquad (\lambda \in S) .$$

Therefore $\hat{h}^2 | S \setminus S_1 = 0$ and $\hat{h}^2(e^{ix_1}) = \overline{\hat{h}^1(1)} = h^1(1)$. Define h in $W(T)$ by

$$h = h^1 + h^2.$$

Then $\hat{h} | S \setminus S_1 = 0$ and $\hat{h}(e^{ix_1}) = 2\hat{h}^1(1) \neq 0$. Also,

$$h_n = h_n^1 + \overline{h_n^1}$$

is real for every n .

Similarly, there is a sequence $k = \{k_n\}$ in $W(T)$, with k_n real for all n , such that $\hat{k} | S \setminus S_2 = 0$ and $\hat{k}(e^{ix_2}) = \hat{k}(-1) \neq 0$. Then $(hk)^\wedge = \hat{h}\hat{k} = 0$, and so $hk = 0$ by the semi-simplicity of $W(T)$. Hence h and k satisfy conditions (i) - (iii).

Case (b). Suppose that $x_1 = 0$ and $x_2 < \pi$. Let a , b be real numbers such that $0 < a < x_2 < b < \pi$, and put

$$S_1 = \{e^{ix} : |x| < a\} , \quad S_2 = \{e^{ix} : a < x < b\} .$$

As in case (a), there is a sequence $h = \{h_n\}$ in $W(T)$, with h_n real for all n , such that $\hat{h} | S \setminus S_1 = 0$ and $\hat{h}(e^{ix_1}) = h(1) \neq 0$. By the complete regularity of $W(T)$, there is a sequence $k^1 = \{k_n^1\}$ in $W(T)$ such that $\hat{k}^1 | S \setminus S_2 = 0$ and $\hat{k}^1(e^{ix_2}) \neq 0$. Let k^2 be the sequence with $k_n^2 = \overline{k_n^1}$. Then k^2 belongs to $W(T)$. Also,

$$\hat{k}^2(e^{ix_2}) = \overline{k^1(e^{-ix_2})} = 0 ,$$

and

$$\hat{k}^2 | (S \setminus \overline{S}_2) = 0 ,$$

where $\overline{S}_2 = \{\overline{z} : z \in S_2\}$. Define $k = k^1 + k^2$ in $W(T)$. Then

$$k_n = k_n^1 + \overline{k_n^1}$$

is real for every n . Also,

$$\hat{k} | S \setminus \{S_2 \cup \overline{S}_2\} = 0 ,$$

and

$$\hat{k}(e^{ix_2}) = \hat{k}^1(e^{ix_2}) \neq 0 .$$

Therefore $\hat{h}\hat{k} = 0$, and hence $hk = 0$. Thus h and k satisfy conditions (i) - (iii).

<u>Case (c)</u>. Suppose that $0 < x_1 < x_2 < \pi$. Let a , b , c be real numbers such that $0 < a < x_1 < b < x_2 < c < \pi$, and put

$$S_1 = \{e^{ix} : a < x < b\} , \quad S_2 = \{e^{ix} : b < x < c\} .$$

As in the construction of the sequence k in case (b), we can find sequences h and k in $W(T)$, with h_n and k_n real for all n , such that

$$\hat{h}(e^{ix_1}) \neq 0 , \quad \hat{h} | S \setminus \{S_1 \cup \overline{S}_1\} = 0 ,$$
$$\hat{k}(e^{ix_2}) \neq 0 , \quad \hat{k} | S \setminus \{S_2 \cup \overline{S}_2\} = 0 .$$

Therefore $\hat{h}\hat{k} = 0$, and so $hk = 0$. Hence h and k satisfy conditions (i) - (iii).

Suppose that h and k are sequences in $W(T)$ satisfying (i) - (iii). Since h_n and k_n are real for all n ,

$$U_h = \sum_{-\infty}^{\infty} h_n T^n \ , \quad U_k = \sum_{-\infty}^{\infty} k_n T^n$$

define bounded linear operators on X . Also,

$$U_h U_k = U_{hk} = 0 \ ,$$

since $hk = 0$. Further, the extensions of U_h and U_k to X_c are

$$(U_h)_c = \sum_{-\infty}^{\infty} h_n (T_c)^n \ , \quad (U_k)_c = \sum_{-\infty}^{\infty} k_n (T_c)^n \ .$$

Now, $e^{ix_1} \in Sp(T) = Sp(T_c)$. Therefore, by Lemma (4.1.5),

$$0 \neq \hat{h}\,(e^{ix_1}) \in Sp\{(U_h)_c\} = Sp(U_h) \ .$$

Hence $U_h \neq 0$. Similarly, $\hat{k}\,(e^{ix_2}) \neq 0$ implies that $U_k \neq 0$. Let

$$Y = cl\{U_k X\} \ .$$

As in the proof of Theorem (4.1.8), it is now elementary to show that Y is a non-trivial ultra-invariant subspace for T .

The algebra $W(T)$ is determined by the magnitude of the numbers $\|T^n\|$. Therefore the condition for complete regularity given by Theorem (4.3.3) is applicable when T acts on a real space. Combining this result with Theorem (4.4.1), we immediately obtain the following real analogue of Theorem (4.3.4).

(4.4.2) Theorem. Let T be an invertible bounded linear operator on a real Banach space. Suppose that $Sp(T)$ contains two points which are not complex conjugates of each other, and that there is a log-convex sequence $\{a_n\}_{n \geq 0}$ such that

$$\text{(i)} \quad \|T^n\| \leq a_{|n|} \qquad (-\infty < n < \infty) \ ,$$

$$\text{(ii)} \quad \sum_{n=1}^{\infty} \frac{\log a_n}{n^2} < \infty .$$

Then T has a non-trivial ultra-invariant subspace.

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